Fall 2010

Optimum sensor placement for source localization and monitoring from received signal strength

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OPTIMUM SENSOR PLACEMENT FOR SOURCE LOCALIZATION AND MONITORING FROM RECEIVED SIGNAL STRENGTH

by

Stella-Rita Chioma Ibeawuchi

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Electrical and Computer Engineering
in the Graduate College of
The University of Iowa

December 2010

Thesis Supervisor: Professor Soura Dasgupta

ABSTRACT

The problem of source localization has become increasingly important in recent years. In source localization, we are interested in estimating the location of a source using various relative position information. This research considers source localization using relative position information provided by Received Signal Strength (RSS) values under the assumption of log-normal shadowing. We investigate an important aspect of source localization, namely, that of optimally placing sensors.

Two specific issues are investigated. The first is one of source monitoring. In this, one must place sensors around a localized source in an optimum fashion subject to the constraint that sensors are at least a certain distance from the source. The second is sensor placement for source localization. In this problem, we assume that the source is uniformly distributed in a circular region. The sensors must be placed in the complement of a larger concentric circle, to optimally localize the source.

The monitoring problem is considered in N-dimensions. The localization problem is in 2-dimensions. The technical problem becomes one of investigating the underlying Fisher Information Matrix (FIM) for optimal monitoring and its expectation for optimal localization. The underlying problem then becomes one of placing sensors to maximize the determinant or the minimum eigenvalue of FIM (or its expectation) or minimize the trace of the inverse of the FIM (or its expectation).

Abstract Approved: ______________________________

Thesis Supervisor

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Title and Department

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Stella-Rita Chioma Ibeawuchi

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Electrical and Computer Engineering at the December 2010 graduation.

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TO GOD BE THE GLORY!

ACKNOWLEDGMENTS

I would like to express the deepest appreciation to my thesis advisor, Professor Soura Dasgupta, who has the attitude and the substance of a genius. Without his guidance and persistent help this dissertation would not have been possible.

I would also like to thank my committee members, Professor ErWei Bai, Professor Mark Andersland, Professor Zhiqiang Liu, Professor Edwin Dove and Professor Raghuraman Mudumbai, for their insightful ideas and sugestions that immensely helped me in accomplishing this work. In recognition of all your help and support, I would like to mention the faculty and staff of the Department of Electrical and Computer Engineering in particular Professor Karl Lonngren and Cathy Kern.

The writing of this desertation has been one of most significant academic challenges I have ever had to face. Without the support and prayers of my parents, brothers and sister, relatives, and friends, this study would not have been completed. It is to them I owe my deepest gratitude.

To my co-workers at the Hospital Stores, University of Iowa Hospitals and Clinics, especially Thomas Furlong, Gary Hagen, Brian Morrison, Tom Seydel and Garth Livingston, for your goodness and for giving me such a pleasant work experience and having confidence in me as well as supporting my research and academic activities at the College of Engineering.

Last but certainly not least, Thanks be to God! You have made my life more bountiful. May Your Name be Exalted, Honored, and Glorified. AMEN.

ABSTRACT

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Two specific issues are investigated. The first is one of source monitoring. In this, one must place sensors around a localized source in an optimum fashion subject to the constraint that sensors are at least a certain distance from the source. The second is sensor placement for source localization. In this problem, we assume that the source is uniformly distributed in a circular region. The sensors must be placed in the complement of a larger concentric circle, to optimally localize the source.

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CHAPTER 1

INTRODUCTION

Source localization is the estimation of the position of a source given the signals received at a sensor array. Of interest is the placement of sensors in such a manner as to accurately estimate the position of the source in an optimum manner.

There are situations in which the sensor locations with respect to the source being observed are determined by constraints imposed by the task. For instance, the sensors may be placed along the periphery of a geographical area or they may be constrained to lie along a line segment. It may also be that the distances between the sensors and the source have to be no less than a certain distance to prevent sensor damage.

The Cramer-Rao Lower Bound (CRLB) is a lower bound on estimate variance that provides a gauge of source position estimator accuracy. This research is focused on the optimal placement of sensors for source localization and monitoring from Recieved Signal Strength (RSS) under log-normal shadowing to obtain the best CRLB.

Terms such as RSS and log-normal shadowing will be explained in the sequel. For the moment, we note that source monitoring is a problem related to source localization. Localization involves estimating the position of a source. Monitoring involves continuing to estimate the position of the source after the initial estimate has been obtained, i.e, to monitor for instances of potential source movement.

1.1 Literature Review

1.1.1 Source Localization

Source localization, in recent times, has assumed increasing interest, and has become the subject of much research. To estimate the position of a source given signals received at an array of sensors is a challenging task that needs to be addressed.

Source localization can be defined as the use of a set of sensors to estimate the precise location of a source based on distinct details or information that are related to the sensors' relative position to the source. The importance of source localization cannot be overemphasized since it is required in several applications such as radar, sonar, mobile wireless communications, radio astronomy, seismology, acoustics, geophysics, wireless sensor networks, to name just a few.

Given a wireless sensor network, position-awareness at the sensors are important for the successful network. Another example of the use of source localization is in cellular wireless communications networks where the base stations have to be able to estimate the location of a mobile user transmitting within its geographical coverage area.

The accuracy of source localization can be evaluated based on the closeness of the estimated position of the source to its real position. This is usually expressed in the Euclidean distance between the real and estimated distances.

Various algorithms have been proposed to estimate the location of a source. Localization techniques depend on the information available at the sensor nodes. This information could be power-level information, Time Difference of Arrival (TDOA), Time of Arrival (TOA), Angle of Arrival (AOA), Time of Flight (TOF), Bearing information, Received Signal Strength (RSS) and Range Measurement.

TOA, TDOA, and AOA usually provide the best results but they are often difficult to obtain since they require a good synchronization between timers (in TOA), exchanges between sensors (in TDOA), or multiple antennas (in AOA). RSS at a given sensor is always easily obtained, some knowledge of the decay rate of the RSS (path loss exponent) is needed for efficient least square estimation when using RSS. However, the most commonly used approaches are the Time Difference of Arrival (TDOA) and distance measurements obtained using RSS.

Given a set of vectors $x_1, x_2..., x_n$, usually 2-D or 3-D vectors with $n \geq 3$ for 2-D and $n \geq 4$ for 3-D, an unknown vector y^* can be estimated from the measured distance of a known vector y and the set of vectors $x_1, x_2..., x_n$ i.e. $d_i = \|x_i - y\|$ where y represents the position of the source, $x_1, x_2..., x_n$ represent the positions of the n sensors. In order to estimate the location of a source using distance measurements, at least 3 non-colinearly situated sensors are needed for 2-D source localization. Similarly for 3-D source localization, at least 4 non-coplanar sensors are needed. The RSS approach involves sending out signal with known strength and using the received signal strength and path loss coefficient to estimate the distance between the source and sensor.

The received signal strength is inversely proportional to the distance from the source. Thus, signal strength depends on the medium of transmission and the signal intensity

$$s = \frac{A}{d^{\beta}} \tag{1.1}$$

where s is the RSS, A is the source signal strength at a unit distance from the source, d is the distance between the sensor and the source and β the path loss coefficient. Suppose y is uniquely specified by d, then y^* can be estimated by the use of linear algorithms, but this approach is not recommended because with noisy measurements of distances, the linear algorithm can provide highly inaccurate estimates even when the noise is small. Several research papers adopt non-linear approaches, involving working with the RSS at the various sensors and then choose y that minimizes the cost function

$$J(y) = \sum_{i=1}^{N} \left[s_i - \frac{A}{\|x_i - y\|^{\beta}} \right]^2 \tag{1.2}$$

where x_i is the location of the ith sensor and s_i is the RSS at the ith sensor [21].

One disadvantage of using RSS as a ranging measurement is that the path loss coefficient is dependent on the transmitting environment. Another problem with this approach is that the greater the source-sensor (transmitter-receiver) distance, the weaker the RSS and also multipath fading can lead to propagation models where the fading becomes sensitive to changes in the distance between source and sensor. To account for the uncertainty in β, one employs the log-normal shadowing paradigm.

Specifically (1.1) is replaced by

$$log\, s_i = log\, A - \beta\, log\, \|x_i - y\| + w_i \tag{1.3}$$

where w is the zero mean Gaussian with variance σ^2. Effectively σ reflects the uncertainty in estimating β.

TDOA-based techniques are more commonly used in passive source localization because they do not require a time stamp when the signal is transmitted, and can achieve relatively good location accuracy. Source localization techniques that use TDOA play important roles in many applications like navigation, localization and tracking of acoustic sources, and location services in mobile communication. The estimation of a source location using TDOA is not an easy task because it involves the use of a set of non-linear equations that is related to the source location and the TDOA measurements. The TDOA technique requires the knowledge of the precise positions of the sensors. A signal is generally subjected to attenuation or path-loss as it propagates over a medium, and then received at a number of different separated sensors. The time-delay depends on the distance between the source and its sensors. Both the time delay and the signal strength information are available when the source signal is captured at the sensors.

Source localization involving TDOA/RSS measurements require a very precise knowledge of the sensors' location since small errors in the sensor location can lead to significant decrease in the accuracy of the source location. A robust algorithm is needed to improve the source localization performance.

[19] suggested the use of TDOA to find an accurate location of a source with the use of information about the signals received at the sensor when the sensors have random errors in their positions. They made use of a weighted matrix that consisted of the sensor position errors to improve the estimation of the source localization. This method uses the Weighted Least Square Minimization that does not have the common convergence and initialization problems.

[26] also focused on using TDOA to estimate source localization in the presence of sensor position errors. Their proposed solution was based on the weighted least square minimization. They estimated the source location with the assumption that the sensor positions are without errors and they used the estimated source location to reduce sensor position noise through estimation process. The source position is estimated again using the improved sensor position. The hope is that the source location estimate improves with each said iteration. Although this process is iterative, its convergence is insensitive to the noise powers in the sensor.

1.1.2 Applications of Source Localization

One of the main applications of source localization is the surveillance and protection of military, industrial areas and densely populated areas. Source localization techniques can be applied in signal processing for wireless communications [38], [40] such as array signal processing and source/sensor localization. Source localization applications can also be found in sonar, radar [6], [46] microphone arrays

[32], wireless sensor networks [7], RFID location systems [31], etc. Researches in the RFID-Assisted Localization and Communication for First Responders project determines the likelihood of using RFID-assisted localization in combination with an *ad-hoc* wireless communication network to provide reliable tracking of first responders in stressed indoor RF environments, where GPS-based localization and links to external communication systems are known not to be reliable.

Acoustic sound localization is the estimation of an acoustic source location given measurements of the sound field at various locations [32]. Microphone arrays are typically used to sample the sound field. The localization of various acoustic sources has many possible application areas, for e.g. voice enhancement, intruder detection, sniper localization, automatic tracking of speakers in an e-conferencing environment, just to mention a few.

Sound source localization applications can be found in radar and sonar localization systems. In sonar signal processing, the focus is on locating underwater acoustic sources using an array of hydrophones. In video conference and multimedia human computer interface applications, microphone arrays have been developed to locate and track speakers head. Recently, microphone arrays have been used in the enhancement of the SNR for speech signal, sound source localization, echo removal, speech recognition and hearing aids.

Microphone arrays focus on different applications e.g. voice input in automobile, hearing aids, desktop PC, teleconferencing, etc, but different applications require different standards such as cost, size, robustness both in the algorithm and the computational requirement, high accuracy, noise and echo cancellation in a room environment [32]. Acoustic signatures are used to estimate vehicle locations in an open-field sensor network [42].

Due to the emergence of small, low power devices that incorporate micro sensing and actuation to wireless communication, there has been much research interest in wireless distributed sensor networks. Sensor networks have various applications including seismic remote sensing, environmental monitoring, underwater acoustics, battlefield surveillance, electronic warfare, and geophysics [23].

These sensor networks are designed to perform functions such as localization, classification, detection and tracking of one or more sources in the sensors field. The sensors are battery powered and have a limited wireless communication bandwidth. Hence, one requires signal processing algorithms that consume less energy and occupy less bandwidth. Source localization, being a signal processing task, is usually carried out by the sensors using a passive and stationary sensor network. The main objective is to estimate the position of moving or stationary sources within the sensor field being monitored by the sensor network [41].

1.2 Related Work

Reference [29] optimized the sensor placement for mobile sensor networks by proposing a motion coordination algorithm that directs the mobile sensor network to an optimum sensor placement. In their work, the assumption of Gaussian white noise measurements with diagonal correlation is made and they presented closed-form expressions for the determinant of the Fisher Information Matrix(FIM) for range-measurements models in non-random scenarios for 2-D and 3-D state spaces. Observe that the diagonal elements of the inverse of the FIM provide the Cramer Rao Lower Bound (CRLB) of the underlying estimation problem.

[24] offers a good overview of existing sensor placement procedures for active Ultra Wideband Sensors. They optimized sensor placement by using range measurements and incorporated PEB (positive error bounds), a lower bound on localization accuracy, to measure the quality of their sensor placement configuration. PEB was derived by [24] using the information inequality for an indoor localization system using Ultra-Wideband (UWB) ranging sensor. They similarly developed RELOCATE, an iterative algorithm, that provided sensor placements so as to minimize PEB at that point. The RELOCATE algorithm is a coordinate descent algorithm that minimizes the PEB one coordinate at a time until convergence occurs. They showed that RELOCATE was optimal and efficient when the range measurements were unbiased and had constant variances. They also showed that careful planning of the sensor placement lead to the use of fewer sensors to achieve the same accuracy than distributing sensors evenly on the area boundary. [24] stated in their work that PEB was not only a theoretical lower bound, but it could also be closely approximated by a maximum likelihood estimator.

[33] proposed a procedure for placing acoustic sensors in 3D space using passive source localization. The average CRB for a surveillance area was minimized with respect to the sensors positions. CRLB in the case of passive source localization

depends on the source/sensor positions as well as on the propagation speed and the assumptions made about the disturbance noise. The form of information used in their research was TDOA and they assumed a zero mean Gaussian noise with constant variance. Simulations showed that while performance of estimation was increased, there was no increment in the number of microphones used, infact, the number could be reduced.

[48] derived properties of the Cramer-Rao bound and designed optimum sensor arrays that minimize CRB for 2D and 3D localization from information gotten from TDOA with the assumption of white measurement noise with uniform covariance. Since the CRB is a square matrix, they considered the minimization of the trace of the CRB.

1.3 The Broad Problem

As noted earlier, our goal is to study optimum sensor placement for source monitoring and localization. The setting we adopt is as follows; we consider a source located at y, and sensors at $x_1, x_2 ..., x_n$, where x_i, y are in R^2 or R^3. The signal model is

$$log\, s_i = log\, A - \beta\, log\, \|x_i - y\| + w_i \tag{1.4}$$

where A and β are known, s_i is the RSS at the ith sensor, and w_i are mutually uncorrelated zero mean Gaussian noise.

In the source monitoring problem, it is assumed that y is known, and the objective is to select the x_i to optimize the underlying FIM in the sense to be described below. In source localization, we assumed a distribution on y, and place x_i to optimize the expected value of the FIM.

To be specific, recall that CRLB matrix is the inverse of the FIM. Thus, optimization involves maximizing certain attributes of the FIM. Three such attributes are considered, these are

(i)Maximization of determinant of the FIM

(ii)Maximization of the smallest eigenvalue of the FIM

(iii)Minimization of the trace of the inverse of the FIM.

It is noteworthy that the last criteria represent minimization of the total mean square localization error. In the localization problem, the FIM in (i-iii) is replaced by the expectation of the FIM.

1.4 Outline

Chapter 2 derives the FIM, states the precise optimization problem for source monitoring and shows that the optimum is achieved if and only if the FIM is a scaled identity matrix.

Chapter 3 discusses how the $x_i \in R^N$, N=2, 3 can be chosen to achieve this optimality condition. Chapter 4 describes how the necessary and sufficient condition for the optimum solution can be met for $n > 2$ and extends these results to arbitrary $N \geq 2$.

In chapter 5, we will formulate the optimal source localization problem and provide a necessary and sufficient condition for the optimum solution, for N=2.

CHAPTER 2

SOURCE MONITORING

In this chapter, we formulate the optimal source monitoring problem and provide a necessary and sufficient condition for the optimum solution. Section 2.1 provides the signal model, i.e., RSS under lognormal shadowing and derives the underlying FIM. Section 2.2 makes precise the optimum source monitoring problem. Section 2.3 provides a necessary and sufficient condition on the FIM for optimality to be achieved.

2.1 The Fisher Information Matrix

For the sake of generality, the results of this chapter are in the general N-dimensional space, though the immediate utility is for $N = 2$ and 3. The signal model involves y and $x_1, x_2..., x_n$, each in R^N, where y is the source location and x_i the location of the i^{th} sensor.

The underlying observations are s_i, the RSS at the i^{th} sensor.

$$log\, s_i = log\, A - \beta\, log\, \|x_i - y\| + w_i \tag{2.1}$$

We assume that with $w_i \approx N(0, \sigma^2)$ mutually uncorrelated, where A and β are known. The underlying estimation problem is that of estimating y from s_i, A, β, x_i and σ^2. We now derive the FIM for this problem, call $L = [log\, s_1, ...log\, s_i]^T$,

observe that the conditional probability density function (pdf) under (2.1) is

$$p_{s|y}(s|y) = \frac{1}{(2\pi)^{n/2}\sigma^n} exp\left\{ -\frac{1}{2}\frac{\|(L-Z)\|^2}{\sigma^2} \right\} \tag{2.2}$$

where $Z = [log\, A - \beta\, log\, \|x_1 - y\|\,, ..., log\, A - \beta\, log\, \|x_n - y\|]^T$.

Thus,

$$X = \frac{\partial}{\partial y} ln p_{s|y}(s|y)$$

$$= -\left[\frac{\partial Z}{\partial y}\right]\left(\frac{L-Z}{\sigma^2}\right)$$

$$= -\frac{1}{\sigma^2}\left(\frac{\partial Z^T}{\partial y}\right) W \tag{2.3}$$

where $W = [w_1, ...w_n]$.

Further, with $z_i = log\, A - \beta\, log\, \|x_i - y\|$,

$$\frac{\partial z_i}{\partial y} = -\frac{\beta}{2}\frac{\partial}{\partial y} log\, \|x_i - y\|^2$$

$$= -\frac{\beta}{ln\, 10}\frac{x_i - y}{\|x_i - y\|^2} \tag{2.4}$$

Thus with

$$U = \left[\frac{x_i - y}{\|x_1 - y\|^2}, \ldots\ldots, \frac{x_i - y}{\|x_n - y\|^2} \right]$$

in (2.3)

$$X = \frac{\beta}{\sigma^2 ln\, 10} UW \tag{2.5}$$

By [43], the FIM is $E\left[XX^T\right]$

$$= \frac{\beta^2}{\sigma^2 (ln\, 10)^2} UU^T \tag{2.6}$$

$$= \frac{\beta^2}{\sigma^2 (ln\, 10)^2} \sum_{i=1}^{n} \frac{(x_i - y)(x_i - y)^T}{\|x_i - y\|^4} \tag{2.7}$$

Observe that the optimization of the FIM is equivalent to the optimization of the matrix F below,

$$F = \sum_{i=1}^{n} \frac{(x_i - y)(x_i - y)^T}{\|x_i - y\|^4} \tag{2.8}$$

2.2 The Optimum Source Monitoring Problem

We now formulate the precise source monitoring problem. As noted above, in the sequel, we will work in general R^N. When we say vectors in R^N are noncoplanar, we mean that for $N = 2$, they are non-collinear.

In source monitoring, the underlying assumption is that y is known. Accordingly, the problem we pose is given y, a scalar d and n, choose x_i to optimize F, such that, for all i,

$$\|x_i - y\| \geq d \tag{2.9}$$

The motivation of this constraint is as follows, should the source represent a threat, then, it maybe desirable for the sensors to maintain a minimal distance from it.

Now, the diagonal elements of the inverse of the FIM provide the Cramer-Rao Lower Bound (CRLB) for the underlying estimation problem. Thus as is typical in detection theory, optimality consitutes either the maximization of the minimum eigenvalue or the determinant of the FIM, or for reasons laid out below, the minimization of the trace of F^{-1}. Accordingly we pose the three problems below.

Problem 1: For a given integer $n \geq N+1$, scalar positive d and $y \in R^N$, find distinct, non-coplanar $x_i \in R^N$, $i \in \{1, \cdots n\}$, such that $\lambda_{\min}(F)$ is maximized subject to (2.9).

Problem 2: For a given integer $n \geq N+1$, scalar positive d and $y \in R^N$, find distinct, non-coplanar $x_i \in R^N$, $i \in \{1, \cdots n\}$, such that $\det(F)$ is maximized subject to (2.9).

Problem 3: For a given integer $n \geq N+1$, scalar positive d and $y \in R^N$, find distinct, non-coplanar $x_i \in R^N$, $i \in \{1, \cdots n\}$, such that $\operatorname{trace}(F^{-1})$ is minimized subject to (2.9).

First note that noncoplanarity is necessary for localization to be possible. It is in principle possible for the FIM to be nonsingular even if the x_i are coplanar.

Observe, the minimization of $trace(F^{-1})$ is equivalent to the minimization of the total mean-square error. Note while problems 1 and 2 were addressed in [11] in 2-dimensions, problem 3 was not addressed at all.

A subtle point concerning this optimization is as follows. Observe that in (2.1) the data has an affine dependence on the Gaussian random variables w_i, but a *nonaffine dependence on y, the vector to be estimated.* Thus, [20], no efficient estimate of y exists from s_i, β, A and σ.

What then is the virtue of considering Problems 1 to 3? Recall the underlying setting is the monitoring of y from the x_i. This will require the x_i to repeatedly acquire the RSS values. Assume that the noise w_i from one sample to the next are iid. Then the Maximum Likelihood estimate under mild regularity conditions is asymptotically efficient. One can show that in the present setting these regularity conditions are indeed met. Thus assuming Maximum Likelihood estimation using data accumulated over time, minimizing the CRLB is very useful.

2.3 A Necessary and Sufficient Condition

First observe from [11] that the inequality constraint in (2.9) is easily replacable by an equality constraint. This is seen by the repeated application of the Lemma below that was partially proved in [11]. This Lemma shows that by scaling down $x_i - y$, one increases (or certainly does not decrease) both the minimum eigenvalue and the determinant, and decreases (or does not increase) the trace of the inverse.

Lemma 2.1. *Consider F in (2.8) under (2.9), $x_i, y \in R^N$, $n > N$, and*

$$G = \frac{vv'}{\|v\|^4} + \sum_{i=2}^{n} \frac{(x_i - y)(x_i - y)'}{\|x_i - y\|^4}, \tag{2.10}$$

where for some scalar α

$$v = \alpha(x_1 - y),$$

and

$$\|v\| = d.$$

Then

$$\lambda_{\min}(F) \leq \lambda_{\min}(G), \tag{2.11}$$

$$\det(F) \leq \det(G), \tag{2.12}$$

and

$$trace(F^{-1}) \geq trace(G^{-1}). \tag{2.13}$$

Proof. It is readily seen that for some $q \in R^2$

$$G = F + qq'.$$

Thus (2.11) clearly holds. Now suppose $\det(F) = 0$. Then as v is a scaled version of $x_1 - y$, $\det(G) = 0$ as well. Thus consider nonsingular F, i.e F is positive definite, and $\det(F) > 0$. Then

$$\begin{aligned} \det(G) &= \det(F)\left(1 + q'F^{-1}q\right) \\ &\leq \det(F). \end{aligned}$$

where the last inequality follows from the fact that as F is positive definite so is F^{-1}. Implicit in (2.13) is the assumption that F is positive definite, and $\det(F) > 0$. Then

$$G^{-1} = F^{-1} - \frac{F^{-1}qq'F^{-1}}{1 + q'F^{-1}q}. \tag{2.14}$$

As F and hence F^{-1} is positive definite it follows that $F^{-1} \geq G^{-1}$. Hence (2.13) follows. $\square$

With

$$z_i = \frac{x_i - y}{d}$$

define

$$Z = \sum_{i=1}^{n} z_i z_i'. \tag{2.15}$$

Then without loss of generality to consider problems 1 to 3 are equivalent to the three problems below, with $N = 3$.

Problem 1A: For a given integer $n \geq N + 1$, find distinct, unit norm $z_i \in R^N$, $i \in \{1, \cdots n\}$, such that $\lambda_{\min}(Z)$ is maximized and the z_i do not lie on an N-dimensional hyperplane.

Problem 2A: For a given integer $n \geq N + 1$, find distinct, unit norm $z_i \in R^N$, $i \in \{1, \cdots n\}$, such that $\det(Z)$ is maximized and the z_i do not lie on an N-dimensional hyperplane.

Problem 3A: For a given integer $n \geq N + 1$, find distinct, unit norm $z_i \in R^N$, $i \in \{1, \cdots n\}$, such that $trace(Z^{-1})$ is minimized and the z_i do not lie on an N-dimensional hyperplane.

Observe that when the z_i have unit norm then

$$\text{trace}(Z) = n. \tag{2.16}$$

We next present a necessary and sufficient condition for the solutions of all three problems *without the non-coplanarity requirement.*

Theorem 2.2. *Consider for an integer $n \geq N + 1$ any $N \times N$ symmetric positive* *definite matrix B, with*

$$trace(B) = n. \tag{2.17}$$

Then

$$\lambda_{\min}(B) \leq \frac{n}{N}, \tag{2.18}$$

$$\det(B) \leq \left(\frac{n}{N}\right)^N, \tag{2.19}$$

and

$$trace(B^{-1}) \geq \frac{N^2}{n}. \tag{2.20}$$

Further equality holds in these inequalities iff

$$B = \frac{n}{N} I. \tag{2.21}$$

Proof. Call $\lambda_i(B)$ the eigenvalues of B. Then we will first show that (2.18-2.20) hold, and that the equality in these hold iff for all $i, j \in \{1, \cdots, n\}$,

$$\lambda_i(B) = \lambda_j(B) = \frac{n}{N}. \tag{2.22}$$

To this end observe that as B is symmetric positive definite and (2.17) holds, the first set of equalities in (2.22) implies the last. Note (2.18) follows from the fact that the minimum eigenvalue of a $N \times N$ symmetric positive definite matrix is no greater than $1/N$-th of its trace, and that it achieves this bound iff (2.22) holds

. Further from Hadamaard's inequality [28], and the fact that the arithmetic mean is greater than or equal to the geomeric mean with equality iff all numbers whose mean they are are the same, one obtains

$$\begin{aligned} [\det(B)]^{1/N} &= \left[\prod_{i=1}^{N} \lambda_i(B)\right]^{1/N} \\ &\leq \frac{\sum_{i=1}^{N} \lambda_i(B)}{N} \\ &= \frac{n}{N}. \end{aligned}$$

Again equality holds iff (2.22) holds. Similarly,

$$\begin{aligned} \frac{trace(B^{-1})}{N} &= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(B)} \\ &\geq \left[\prod_{i=1}^{N} \frac{1}{\lambda_i(B)}\right]^{1/N}, \end{aligned}$$

with equality holding iff (2.22) holds. Further under (2.22), (2.20) must hold.

Now we prove that only symmetric positive definite matrix B for which (2.22) holds, is as in (2.21). To this end we observe that under (2.22)

$$\lambda_{min}(B) = \frac{n}{N}. \tag{2.23}$$

Consequently:

$$A = B - \frac{n}{N} I \geq 0. \tag{2.24}$$

Because of (2.17) either all diagonal elements of B are n/N or at least one is less than n/N. In the latter case A has a negative diagonal element and (2.24) cannot hold. In the former case all diagonal elements of A are zero. Consequently (2.24) holds iff $A = 0$. This completes the proof. $\square$

Thus the solution to both problems is characterized by unit norm z_i that result in

$$Z = \frac{n}{N} I. \tag{2.25}$$

2.4 Conclusion

In this chapter, we have given a precise formulation of the optimum source monitoring problem and have shown that with

$$z_i = \frac{x_i - y}{d} \tag{2.26}$$

optimum x_i must be such that $\|z_i\| = 1$ and $\sum_{i=1}^{n} z_i z_i^T$ is a scaled identity matrix. In the next chapter, we will show how to choose unit z_i to satisfy this condition for N=2 and 3.

CHAPTER 3

SOLUTION TO THE SOURCE MONITORING PROBLEM IN 2 AND 3-DIMENSIONS

In chapter 2, we provided a necessary and sufficient condition for optimality of the source monitoring problem.

Specifically, with

$$z_i = \frac{x_i - y}{d} \tag{3.1}$$

we require that $\|z_i\| = 1$, z_i be non collinear in 2D and non-coplanar in 3D and with N=2 and 3 respectively, $n \geq N+1$ there holds

$$Z = \sum_{i=1}^{n} z_i z_i^T = \frac{N}{n} I \tag{3.2}$$

In sections (3.1) and (3.2), we explain how this condition can be met in 2 and 3-dimensions respectively. Section (3.3) provides simulations for 2D.

3.1 Choosing z_i in 2D

We must find for $n > 2$, a 2-dimensional unit z_i for which

$$Z = \frac{n}{2} I. \tag{3.3}$$

We first note certain simple structural issues associated with (2.15). In particular note that for any orthogonal matrix Q, (2.15) is unaltered if z_i are replaced $\pm Qz_i$. In the sequel, when we refer to uniqueness of the z_i for achieving (3.3), we will imply uniqueness to within these equivalences.

There is also no loss of generality in assuming that $z_1 = [1, 0]' = e_1$. We next describe one class of potential solutions for achieving (3.3). Namely with Q an orthogonal matrix, select for $i \in \{1, \cdots, n\}$

$$z_i = Q^{i-1} e_1. \tag{3.4}$$

Then (3.3) is equivalent to:

$$\sum_{i=1}^{n} Q^{i-1} e_1 e_1' Q'^{i-1} = \frac{n}{2} I. \tag{3.5}$$

Since Q is an orthogonal matrix, this in turn is equivalent to

$$\sum_{i=2}^{n+1} Q^{i-1} e_1 e_1' Q'^{i-1} = \frac{n}{2} I. \tag{3.6}$$

Subtracting (3.5) from (3.6), we obtain that for (3.5) to be true, there must hold:

$$Q^n e_1 = \pm e_1. \tag{3.7}$$

Now note that an arbitrary 2×2 orthogonal matrix is completely characterized by the Givens rotation [17]:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Most specifically, with such a Q, Qx is x rotated counter clockwise by θ. Further

$$Q^i = \begin{bmatrix} \cos i\theta & -\sin i\theta \\ \sin i\theta & \cos i\theta \end{bmatrix}.$$

Thus (3.7) becomes:

$$\cos n\theta = \pm 1$$

and

$$\sin n\theta = 0.$$

To within the equivalences noted earlier one thus obtains:

$$z_i = \begin{bmatrix} \cos(\pi(i-1)/n) \\ \sin(\pi(i-1)/n) \end{bmatrix}.$$

Indeed in this case the (1,1) element of Z is

$$\begin{aligned} Z(1,1) &= 1+\sum_{i=2}^{n}\cos^2(\pi(i-1)/n) \\ &= \sum_{i=0}^{n-1}\cos^2(\pi i/n) \\ &= \frac{n}{2}+\mathrm{Re}\left[\sum_{i=0}^{n-1}\exp(j2\pi i/n)\right] \\ &= \frac{n}{2}. \end{aligned}$$

Likewise

$$\begin{aligned} Z(1,2) &= Z(2,1) \\ &= \sum_{i=2}^{n-1}\cos(\pi(i-1)/n)\sin(\pi(i-1)/n) \\ &= \sum_{i=0}^{n-1}\cos(\pi i/n)\sin(\pi i/n) \\ &= \mathrm{Im}\left[\sum_{i=0}^{n-1}\exp(j2\pi i/n)\right] \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} Z(2,2) &= \sum_{i=2}^{n}\cos^2(\pi(i-1)/n) \\ &= \sum_{i=0}^{n-1}\sin^2(\pi i/n) \\ &= \frac{n}{2}-\mathrm{Re}\left[\sum_{i=0}^{n-1}\exp(j2\pi i/n)\right] \\ &= \frac{n}{2}. \end{aligned}$$

Thus a set of n vectors each rotated from their neighbor by π/n, and their equivalences suffice to achieve (3.3). It should be stressed that this choice of z_i is by no means unique for $n > 3$. We now argue that it is unique for $n = 3$, within of course the equivalences noted above. Indeed for $n = 3$, to within equivalences we must choose $z_1 = [1, 0]'$ and $z_i = [\cos\theta_i, \sin\theta_i]'$, for $i \in \{1, 2\}$. Then (3.3) becomes:

$$1 + \cos^2\theta_1 + \cos^2\theta_2 = 3/2 \tag{3.8}$$

$$\sin^2\theta_1 + \sin^2\theta_2 = 3/2 \tag{3.9}$$

$$\cos\theta_1 \sin\theta_1 + \cos\theta_2 \sin\theta_2 = 0. \tag{3.10}$$

Both (3.8) and (3.9) reduce to

$$\cos(2\theta_1) + \cos(2\theta_2) = -1. \tag{3.11}$$

Further (3.10) becomes:

$$\sin(2\theta_1) + \sin(2\theta_2) = 0. \tag{3.12}$$

Defining $x_i = \cos(2\theta_i)$ this pair of equations reduce to:

$$x_1 + x_2 = -1. \tag{3.13}$$

and one of

$$\sqrt{1 - x_1^2} + \sqrt{1 - x_2^2} = 0, \tag{3.14}$$

and

$$\sqrt{1 - x_1^2} - \sqrt{1 - x_2^2} = 0. \tag{3.15}$$

In either case one has,

$$x_1^2 = x_2^2. \tag{3.16}$$

Since $x_1 = -x_2$ will not satisfy (3.13), we have that to within equivalences

$$\cos(2\theta_1) = \cos(2\theta_2) = -1/2,$$

provides a unique solution to within the stated equivalences, e.g.

$$\theta_1 = \pi/3$$

and

$$\theta_2 = 2\pi/3.$$

Observe that z_i are noncollinear. We conclude this section with an observation relating our solutions to spherical codes. Observe to within an equivalence in the n=3 case, one choice of the z_i are depicted in fig 3.1. One way of describing this is that these z_1,

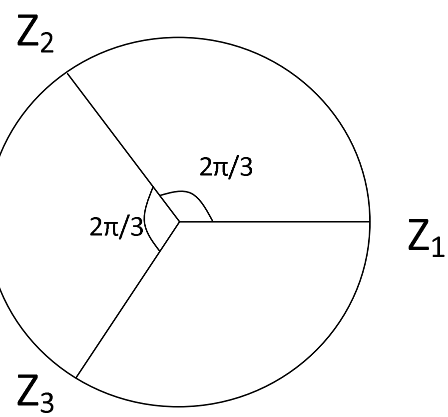

Figure 3.1: Depiction of relation to spherical codes

z_2 and z_3 are a collection of three vectors on the unit circle such that the minimum distance between them is maximized.

All the solutions given in this section, to within an equivalence, have these properties. These are infact characteristics of spherical codes.

3.2 Solution for Source Monitoring in 3D

In this case, we require that

$$Z = \frac{n}{N} I. \tag{3.17}$$

In this section we describe how (3.17) can be met with noncoplanar z_i, for $n > N = 3$. We first note certain simple structural issues associated with (2.15). In particular note that for any orthogonal matrix Q_n, (2.15) is unaltered if z_i are replaced $\pm Q_n z_i$. There is also no loss of generality in assuming that $z_1 = \frac{1}{\sqrt{3}}[1, 1, 1]' = e_1$. We next describe one class of potential solutions for achieving (3.17). Namely for a given n, with $Q_n \in R^{3\times 3}$ an orthogonal matrix, select for $i \in \{1, \cdots, n\}$

$$z_i = Q_n^{i-1} e_1. \tag{3.18}$$

Then, with $N = 3$, (3.17) is equivalent to:

$$\sum_{i=0}^{n-1} Q_n^i e_1 e_1' Q_n'^i = \frac{n}{3} I. \tag{3.19}$$

We further note that an orthogonal Q_n that satisfies (3.19), can be used to determine, an orthogonal P_n that satisfies:

$$\sum_{i=0}^{n-1} P_n^i \eta \eta' P_n'^i = \frac{n}{3} I, \tag{3.20}$$

for any other given unit $\eta \in R^3$. Indeed for such a unit η there exists an orthogonal matrix P, such that

$$\eta = Pe_1.$$

Then:

$$\begin{aligned} \frac{n}{3}I &= \sum_{i=0}^{n-1} Q_n^i e_1 e_1' Q_n'^i \\ &= P \left[\sum_{i=0}^{n-1} Q_n^i P' P e_1 e_1' P' P Q_n'^i \right] P' \\ &= \sum_{i=0}^{n-1} (PQ_nP')^i \eta\eta' (PQ_nP')'^i . \end{aligned}$$

Thus $P_n = PQ_nP'$, satisfies (3.20). There are some notable differences between the two and three dimensional cases. First in two dimensions one can choose Q_n to be a rotation matrix, i.e. one that is orthogonal and has determinant 1. The sequel will demonstrate that such a Q_n in three dimensions will result in z_i that are coplanar. Second, in two dimensions it is impossible to satisfy (3.19) that involve z_i that are in collinear. By contrast in three dimensions it is possible to identify a class of real orthogonal Q_n satisfying (3.19) that involve z_i that are coplanar. Indeed choose:

$$Q_4 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.21}$$

and verify that (3.19) holds. Yet the third element of each z_i is 1, and thus the z_i are coplanar.

We now provide a necessary condition on Q_n to ensure non-coplanarity.

Lemma 3.1. *Consider for $n > 3$, any orthogonal $Q_n \in R^{3\times 3}$, and a unit $v \in R^3$. Suppose:*

$$\sum_{i=0}^{n-1} Q_n^i v v' Q_n'^i = \frac{n}{3} I, \tag{3.22}$$

and $\{v, Q_n v, Q_n^2 v, \cdots, Q_n^{n-1} v\}$ are non-coplanar. Then the eigenvalues of Q_n are, for some $\theta \in \{(0, 2\pi) - \{\pi\}\}$, $\left\{-1, e^{j\theta}, e^{-j\theta}\right\}$.

Proof. We first assert that Q_n cannot have an eigenvalue at 1. To establish a contradiction suppose the contrary is true. Then for some $p \in R^3$,

$$p' Q_n = p'.$$

Thus, for all $i \in \{0, 1, \cdots, n-1\}$,

$$p' Q_n^i v = p' v = \text{ constant }.$$

Consequently $\{v, Q_n v, Q_n^2 v, \cdots, Q_n^{n-1} v\}$ are coplanar. Thus as all eigenvalues of Q_n are on the unit circle, $Q_n \in R^{3\times 3}$, and its complex eigenvalues must be in conjugate pairs, either exactly one eigenvalue of Q_n is -1, or all three are -1. In the latter case as Q_n is orthogonal, $Q_n = -I$, and one would obtain,

$$\sum_{i=0}^{n-1} Q_n^i v v' Q_n'^i = n v v',$$

violating (3.22). Thus exactly one eigenvalue of Q_n is -1 and the other two on the unit circle are complex and form a conjugate pair. This completes the proof. $\square$

Observe that when Q_n has eigenvalues $\{-1, e^{j\theta}, e^{-j\theta}\}$, its determinant is -1, and it cannot be a rotation matrix. We now characterize one class of Q_n that achieves (3.19).

Theorem 3.2. *Suppose $z_1 = [1,1,1]'/\sqrt{3}$ and for $n > 3$ the unitary matrix $T_n \in C^{3\times 3}$ obeys the following:*

(a) $T_n \, diag \left\{e^{j(2\pi/n-\pi)}, e^{-j(2\pi/n-\pi)}, -1\right\} T_n^H \in R^{3\times 3}$.

(b) Each element of $T_n^H z_1$ has magnitude $1/\sqrt{3}$.

Then $Q_n = T_n \, diag \left\{e^{j(2\pi/n-\pi)}, e^{-j(2\pi/n-\pi)}, -1\right\} T_n^H$ is orthogonal and obeys (3.19). Further $\{z_1, Q_n z_1, Q_n^2 z_1, \cdots, Q_n^{n-1} z_1\}$ are noncoplanar.

Proof. That Q_n as defined in the theorem statement is orthogonal is trivial. Call

$$\theta_n = 2\pi/n - \pi. \tag{3.23}$$

Suppose first that $\{z_1, Q_n z_1, Q_n^2 z_1, \cdots, Q_n^{n-1} z_1\}$ are coplanar. Then for some $\rho \in R^3$,

$\rho \neq 0$ and all $i, k \in \{0, \cdots, n-1\}$,

$$\rho' Q_n^i z_1 = \rho' Q_n^k z_1. \tag{3.24}$$

Consequently with $p = \rho' T_n = [p_1, p_2, p_3]' \neq 0$

$$w_n = [w_{1n}, w_{2n}, w_{3n}]' = T^H z_1, \tag{3.25}$$

for all $i, k \in \{0, \cdots, n-1\}$ there holds,

$$\begin{aligned} & p_1 w_{1n} e^{ji\theta_n} + p_2 w_{2n} e^{-ji\theta_n} + (-1)^i p_3 w_{3n} \\ = \quad & p_1 w_{1n} e^{jk\theta_n} + p_2 w_{2n} e^{-jk\theta_n} + (-1)^k p_3 w_{3n}. \end{aligned} \tag{3.26}$$

By hypothesis for all $i \in \{1, 2, 3\}$

$$|w_{in}| = 1/\sqrt{3}. \tag{3.27}$$

By respectively comparing the $\{i, k\}$ pairs, $\{0, 2\}$ and $\{1, 3\}$ (recall $n > 3$), we obtain

$$\begin{bmatrix} 1 & 1 \\ e^{j\theta_n} & e^{-j\theta_n} \end{bmatrix} \begin{bmatrix} \left(e^{j2\theta_n} - 1\right) p_1 w_{1n} \\ \left(e^{-j2\theta_n} - 1\right) p_2 w_{2n} \end{bmatrix} = 0.$$

As $n > 3$, under (3.23) and (3.27), the 2×2 matrix in the left hand side of the above equation is nonsingular, $w_{in} \neq 0$ and $\left(e^{j2\theta_n} - 1\right) \neq 0$. Thus, $p_1 = p_2 = 0$. Thus from (3.26) by using $i = 0$ and $k = 1$, $p_3 = 0$. This establishes a contradiction and $\{z_1, Q_n z_1, Q_n^2 z_1, \cdots, Q_n^{n-1} z_1\}$ are indeed noncoplanar.

To complete the proof we show that with w_n as in (3.25) and (3.27), there holds:

$$\begin{aligned} & \sum_{i=0}^{n-1} diag\left\{e^{ji\theta_n}, e^{-ji\theta_n}, (-1)^i\right\} w_n w_n^H \\ & diag\left\{e^{-ji\theta_n}, e^{ji\theta_n}, (-1)^i\right\} \\ = & \frac{n}{3} I. \end{aligned} \tag{3.28}$$

Then the result will follow because T_n is unitary. That the diagonal elements of the matrix on the left hand side of (3.28) equal $n/3$ is a direct consequence of (3.27). Since the (1,3) element is just the conjugate of the (2,3) element, it suffices to show that the (1,3) and (1,2) elements are zero. Because of (3.23) the (1,2) element on the left hand side of (3.28) equals:

$$\begin{aligned} w_{1n} w_{2n}^* \sum_{i=0}^{n-1} e^{2ji\theta_n} &= w_{1n} w_{2n}^* \frac{1 - e^{2jn\theta_n}}{1 - e^{2j\theta_n}} \\ &= 0. \end{aligned}$$

Similary, the (1,3) element on the left hand side of (3.28) equals:

$$\begin{aligned} w_{1n} w_{3n}^* \sum_{i=0}^{n-1} e^{ji(\theta_n - \pi)} &= w_{1n} w_{3n}^* \frac{1 - e^{jn(\theta_n - \pi)}}{1 - e^{j(\theta_n - \pi)}} \\ &= 0. \end{aligned}$$

The proof is thus complete. □

Generating T_n that meets the requirements of the theorem is straight forward. For example:

$$T_n = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.29}$$

results in

$$Q_n = \begin{bmatrix} \cos(\theta_n) & \sin(\theta_n) & 0 \\ -\sin(\theta_n) & \cos(\theta_n) & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.30}$$

where θ_n is as in (3.23). The theorem of course provides a much wider class Q_n. Observe that unlike the 2D case, these solutions do not represent optimal 3D spherical codes. However, in the n=4 case when (3.30) leads to $z_1, \cdots z_4$ that are vertices of a tetrahedron, the resulting code has been classified as a "good spherical code".

To understand (3.30) for n>4, suppose the first two dimensions represent the x and y axes and third the z-axis. Then multiplication by Q as in (3.30) flips the sign of the z-coordinate and rotates counterclockwise by θ_n parallel to the $x - y$ plane.Thus, as depicted in figure 3.2 for $n = 6$, the points characterized by $z_1, Qz_1, \cdots Q^{n-1}z_1$, lie on the bases of two cones, one the inverted version of the other. The bases are parallel to the $x - y$ axes, and have their rims on the surface of the unit sphere. Each cone has its apex at the origin and is parallel to the z-axis. The points alternate between the bases. Going from one point on the *same* base to its neighbor involves a rotation by $2\pi/n$ parallel to the x-y plane. The first two points on each base are rotated by π/n, parallel to the x-y plane.

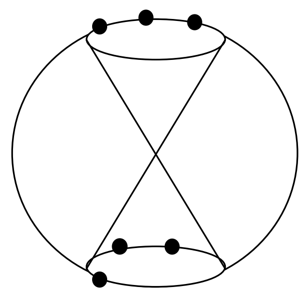

Figure 3.2: Alternation of points between bases of rotated cones for n=6

3.3 Simulations for the 2D case

In the simulations below for the 2D case, $A = 1000$, $\beta = 2$, $\mu = 0.001$, $y = [0.5; 0]$. 1000 runs of σ are used to average the error with plots of mean square error vs σ.

From the simulations of the 2D case, the optimized sensor positions have good performance relative to the arbitrary sensor positions, which have much higher averaged error, thus, the optimization of the sensor positions achieves the lowest possible mean square error and hence, performs an optimal monitoring and localization of the source.

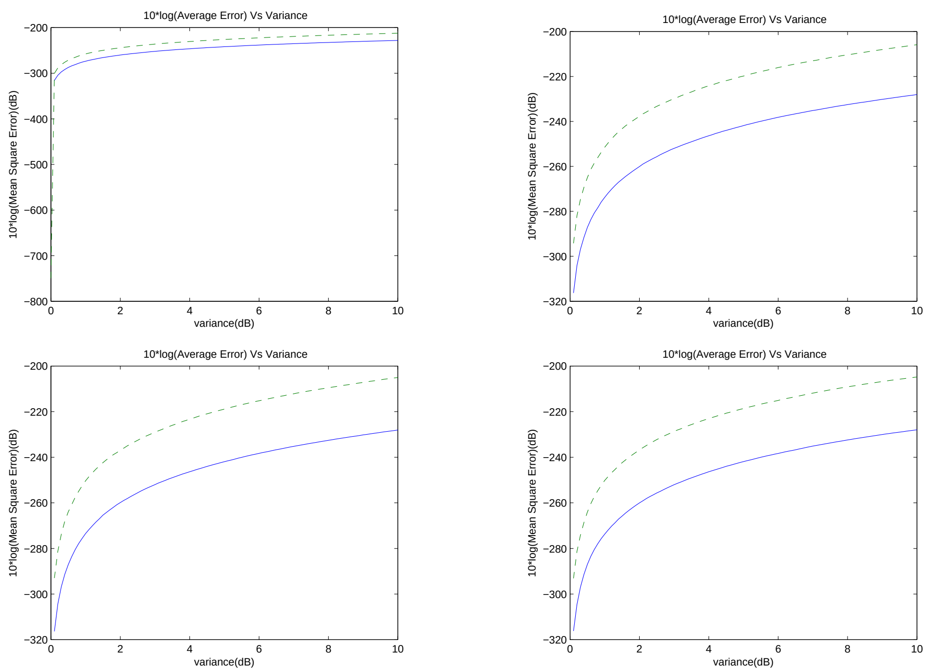

Figure 3.3: Comparison between the Mean Square Error Vs Variance of arbitrary sensor locations and the optimized sensor positions at $\theta = 0, \pi/3, 2\pi/3$ with chosen source location y at $[0.5, 0]'$. Diagram legend: Dashed line-arbitrary sensor position, Straight line-optimized sensor positions.

CHAPTER 4

OPTIMUM SOURCE MONITORING IN N-DIMENSIONS

4.1 Introduction

The previous chapters dealt the optimum source monitoring problem when both the source location y and the sensor locations x_i are either 2-dimensional or 3-dimensional. In this chapter we consider the N-dimensional case where $N > 3$. The solutions we provide here specialize to those provided earlier for the two and three dimensional cases.

The motivation for considering the general N-dimesnsional case is as follows. First it is a natural and potentially useful generalization of the lower dimensional cases. Second, it has potential applications when y represents a vector of features and the x_i reference features. Finally, the solution to this problem in two dimensions represents optimal spherical codes. In the three dimensions it is identical at least for the case of four sensors to a code that has been recognized as a *good spherical code*. Thus the general N-dimensional solution has the potential of also generating good spherical codes.

In a sense the results here are stronger than in the previous chapters. Specifically, recall that one class of solutions presented in the earlier chapters involve the case where with Q an orthogonal matrix, and d the minimum distance that the sensors are allowed to be from the source, for each i, and arbitrary x_1, obeying $\|x_i - y\| = d$,

$$\frac{x_i - y}{d} = Q^{i-1}\frac{x_1 - y}{d}.$$

The previous chapters provide sufficient conditions on how Q can be selected. In this chapter we provide a *necessary and sufficient* condition on Q. Section 4.2 recounts the transformed problem we must solve. Section 4.3 provides a necessary conditions on Q. Sections 4.4 and 4.5 respectively provide the necessary and sufficient condition on Q for even and odd N.

4.2 Preliminaries

In this chapter we assume that there are $n > N$, distinct, $x_i \in \mathbb{R}^N$, and $y \in \mathbb{R}^N$, such that the x_i do not lie in a hyperplane of dimension less than N. As noted in the previous chapter optimality is equivalent to the requirement that with *unit norm*

$$z_i = \frac{x_i - y}{d}$$

and

$$Z = \sum_{i=1}^{n} z_i z_i' \tag{4.1}$$

the z_i do not lie on a hyperplane of dimension less than N, and there holds:

$$Z = \frac{n}{N}I. \tag{4.2}$$

Our focus will be to choose for a given $n > N$, and z_1, orthogonal matrices $Q \in \mathbb{R}^{N\times N}$ such that with

$$z_i = Q^{i-1}z_1. \tag{4.3}$$

the z_i do not lie on a hyperplane of dimension less than N, and (4.2) i.e.

$$\sum_{i=0}^{n-1} Q^i z_1 z_1' Q'^i = \frac{n}{N}I. \tag{4.4}$$

In doing so we observe that as before for any orthogonal matrix Q, (4.1) is unaltered if z_i are replaced $\pm Qz_i$. We also note that it suffices to consider an arbitrary z_1, as should Q satisfy (4.4) with a given z_1 and if $\eta = Pz_1$, with P orthogonal, then (4.4) holds with z_1 and Q respectively replaced by η and PQP'.

4.3 A necessary condition

We first provide a necessary condition on orthogonal Q to ensure that for a given non-zero z_1, and $n > N$, the z_i generated as in (4.3) do not lie on a hyperplane of dimension less than N.

Lemma 4.1. *Consider for $n > N \geq 2$, any orthogonal $Q \in \mathbb{R}^{N\times N}$, and a unit*

$v \in \mathbb{R}^N$. *Suppose:*

$$\sum_{i=0}^{n-1} Q^i v v' Q'^i = \frac{n}{N} I, \tag{4.5}$$

and $\{v, Qv, Q^2v, \cdots, Q^{n-1}v\}$ *do not lie on a hyperplane of dimension less than* N. *Then the eigenvalues of* Q *are simple, on the unit circle, none is at* 1 *and complex eigenvalues appear in conjugate pairs.*

Proof. Since $Q \in \mathbb{R}^{N \times N}$, is orthogonal, all eigenvalues are on the unit circle and complex ones appear in conjugate pairs.

We next assert that Q cannot have an eigenvalue at 1. To establish a contradiction suppose the contrary is true. Then for some $p \in \mathbb{R}^N$,

$$p'Q = p'.$$

Thus, for all $i \in \{0, 1, \cdots, n-1\}$,

$$p'Q_n^i v = p'v = \text{ constant }.$$

Consequently $\{v, Qv, Q^2v, \cdots, Q^{n-1}v\}$ lie on an $(N-1)$-dimensional hyperplane.

Finally we prove that the eigenvalues of Q are distinct. Suppose, to establish a contradiction, an eigenvalue of Q has multiplicity greater than 1. Then for a unitary matrix $U \in C^{N \times N}$ and diagonal $\Lambda \in C^{(N-2)\times(N-2)}$, and scalar, possibly complex, nonzero λ, there holds:

$$Q = U \begin{bmatrix} \Lambda & 0 \\ 0 & \lambda I_2 \end{bmatrix} U^H.$$

Define

$$Uv = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{4.6}$$

where $v_2 \in C^2$. Observe for all $i \in \{0, \cdots, n-1\}$,

$$Q^i v = U \begin{bmatrix} \Lambda^i v_1 \\ \lambda^i v_2 \end{bmatrix}.$$

Thus as U is unitary, (4.5) holds iff the following holds:

$$\begin{bmatrix} \sum_{i=0}^{n-1} \Lambda^i v_1 v_1^H \Lambda^{Hi} & \sum_{i=0}^{n-1} \Lambda^i v_1 v_2^H \lambda^{*i} \\ \sum_{i=0}^{n-1} \lambda^i v_2 v_1^H \Lambda^{Hi} & \left(\sum_{i=0}^{n-1} |\lambda|^{2i}\right) v_2 v_2^H \end{bmatrix} = \frac{n}{N} I.$$

This in particular implies that the matrix $v_2 v_2^H$ that has rank at most 1, is a scaled identity. This leads to a contradiction, thus completing the proof. $\square$

The implication of this result is as follows. For even $N = 2M$ the eigenvalues of Q must be

$$\{e^{\pm j\theta_i}\}_{i=1}^{M} \tag{4.7}$$

where for all $i \neq k$

$$e^{\pm j\theta_i} \neq e^{\pm j\theta_k}, \tag{4.8}$$

and for all $i \in \{1, \cdots, M\}$,

$$e^{j\theta_i} \notin \{-1, 1\}. \tag{4.9}$$

On the other hand for odd $N = 2M + 1$, with θ_i as above the set of eigenvalues of Q must be:

$$\{e^{\pm j\theta_i}\}_{i=1}^{M} \bigcup \{-1\}. \tag{4.10}$$

In the remaining two sections of this chapter, we provide necessary and sufficient conditions on Q for the case of even and odd N.

4.4 The even dimensional case

Having established a necessary condition on Q, we now turn to characterizing all orthogonal real Q, given a unit norm z_1 that obey the required condition, when N

is even. Specifically, we use the fact that every $Q \in \mathbb{R}^{N\times N}$ admits the factorization:

$$Q = T\Lambda T^H, \tag{4.11}$$

where $T \in C^{N\times N}$ is unitary and $\Lambda \in C^{N\times N}$ is diagonal with all elements on the unit circle. The theorem below characterizes the relationship between T, Λ and z_1.

Theorem 4.2. *Suppose for integer $M \geq 1$, $n > N = 2M$, and $z_1 \in \mathbb{R}^N$ is a unit norm vector. Consider an orthogonal $Q \in \mathbb{R}^{N\times N}$ as in (4.11) where $T \in C^{N\times N}$ is unitary and $\Lambda \in C^{N\times N}$ is diagonal with all elements on the unit circle. Then Q satisfies (4.4) and the set of vectors $\{z_1, Qz_1, \cdots, Q^{n-1}z_1\}$ do not lie on an $(N-1)$-dimensional iff the following hold.*

(a) Each element of $T^H z_1$ has magnitude $1/\sqrt{N}$.

(b) There exist real $\theta_1, \cdots, \theta_M$, such that the diagonal elements of Λ are $e^{\pm j\theta_i}$, are distinct and appear in conjugate pairs.

(c) For all $i \in \{1, \cdots, M\}$, $e^{\pm j\theta_i} \neq \pm 1$.

(d) For all $\{i, j\} \subset \{1, \cdots, M\}$, including $i = j$, $\theta_i \pm \theta_j$ are integer multiples of $2\pi/n$.

Proof. We first prove necessity.

The necessity of (b) and (c) follows from Lemma 4.1. Define:

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = T^H z_1. \tag{4.12}$$

Observe (4.4) is equivalent to:

$$\sum_{i=0}^{n-1} \Lambda^i w w^H \Lambda^{Hi} = \frac{n}{N} I. \tag{4.13}$$

The k-th diagonal element of the left hand side is $n\,|w_k|^2$. Hence (a) must hold.

The off diagonal elements on the left hand side of (4.13) are two types. First for suitable $\{k,l\} \subset \{1, \cdots, N\}$, $k \neq l$, and $\{r,s\} \subset \{1, \cdots, M\}$, $r \neq s$,

$$w_k w_l^* \sum_{i=0}^{n-1} e^{\pm ji(\theta_r \pm \theta_s)} = w_k w_l^* \frac{1 - e^{\pm jn(\theta_r \pm \theta_s)}}{1 - e^{\pm j(\theta_r \pm \theta_s)}}. \tag{4.14}$$

Because of (b) and (c) the denominators are non-zero. Thus as (a) holds, for $r \neq s$, $n(\theta_r \pm \theta_s)$ is a multiple of 2π, i.e. $\theta_r \pm \theta_s$ is a multiple of $2\pi/n$.

The second type of off diagonal elements on the left hand side of (4.13) are: for suitable $\{k,l\} \subset \{1, \cdots, N\}$, $k \neq l$, and $r \in \{1, \cdots, M\}$,

$$w_k w_l^* \sum_{i=0}^{n-1} e^{\pm 2ji\theta_r} = w_k w_l^* \frac{1 - e^{\pm j2n\theta_r}}{1 - e^{\pm 2j\theta_r}}. \tag{4.15}$$

Because of (c) the denominators are again non-zero. Thus as (a) holds, $2n\theta_r$ is a multiple of 2π. Thus (d) is also necessary.

To prove sufficiency we first note that to show that (4.4) holds it suffices to show that (4.13) holds. As all elements of Λ are on the unit circle and (a) holds, clearly the diagonal elements of the left hand side of (4.13) are all n/N. Consider next the off diagonal elements of the form in (4.14). Because of (b-d), these are clearly zero, as are the off-diagonal elements represented by (4.15). Thus indeed (4.13) and hence (4.4) hold.

It remains to show that $\{z_1, Qz_1, Q^2z_1, \cdots, Q^{n-1}z_1\}$ do not inhabit an $(N-1)$-dimensional hyperplane. To establish a contradiction suppose they do lie on an $(N-1)$-dimensional hyperplane. Then for some $\rho \in \mathbb{R}^N$, $\rho \neq 0$ and all $i, k \subset \{0, \cdots, n-1\}$,

$$\rho' Q^i z_1 = \rho' Q^k z_1. \tag{4.16}$$

Consequently with $p^H = \rho' T_n \neq 0$, for all $i \in \{0, \cdots, n-2\}$

$$p^H \left(\Lambda^i - \Lambda^{i+1}\right) w = 0. \tag{4.17}$$

Without sacrificing generality assume that

$$\Lambda = diag\left\{e^{j\theta_1}, e^{-j\theta_1}, e^{j\theta_2}, e^{-j\theta_2}, \cdots, e^{j\theta_M}, e^{-j\theta_M}\right\}. \tag{4.18}$$

Define:

$$p^H = [p_1^*, \cdots p_N^*].$$

As $n > N = 2M$, under (4.18) there holds:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ e^{j\theta_1} & e^{-j\theta_1} & e^{j\theta_2} & e^{-j\theta_2} & \cdots & e^{j\theta_M} & e^{-j\theta_M} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{j(N-1)\theta_1} & e^{-j(N-1)\theta_1} & e^{j(N-1)\theta_2} & e^{-j(N-1)\theta_2} & \cdots & e^{j(N-1)\theta_M} & e^{-j(N-1)\theta_M} \end{bmatrix} \begin{bmatrix} p_1^* w_1 \left(1-e^{j\theta_1}\right) \\ p_2^* w_2 \left(1-e^{-j\theta_1}\right) \\ p_3^* w_3 \left(1-e^{j\theta_2}\right) \\ p_4^* w_4 \left(1-e^{-j\theta_2}\right) \\ \vdots \\ p_{N-1}^* w_{N-1} \left(1-e^{j\theta_M}\right) \\ p_N^* w_N \left(1-e^{-j\theta_M}\right) \end{bmatrix} = 0. \tag{4.19}$$

The $N \times N$ matrix on the left hand side of (4.19) is a Vandermonde matrix. Thus because of (b), the matrix is nonsingular. Thus for (4.19) to hold, because of (c), for all $i \in \{1, \cdots, N\}$, $p_i w_i = 0$. Because of (a), this must mean that $p = 0$, establishing a contradiction. □

The foregoing represents a complete characterization of Q when $N = 2M$, and $n > N$. We now show that for every $M \geq 1$, a Q conforming to this characterization can be found. First suppose Λ is as in (4.18). Define:

$$T_e = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}. \tag{4.20}$$

Then the $N \times N$ matrix

$$T = diag\{T_e, T_e, \cdots, T_e\} \tag{4.21}$$

obeys (a) if $z_1 = [1, \cdots, 1]'/\sqrt{N}$.

Indeed in this case with:

$$Q_i = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \tag{4.22}$$

one obtains:

$$Q = diag\{Q_1, Q_2, \cdots, Q_M\}. \tag{4.23}$$

The θ_i chosen to be odd multiples of π/n will satisfy (d). For example one could choose for $k \in \{1, \cdots, M\}$

$$\theta_k = \frac{(2k-1)\pi}{n}. \tag{4.24}$$

Given that there are M of these with $n > 2M$, one has for all $k \in \{1, \cdots, M\}$ that

$$0 < \theta_k \leq \frac{(2M-1)\pi}{n} < \pi, \tag{4.25}$$

i.e. these satisfy (b-c) as well. In the case when $M = 1$, this solution conforms to the solution in the previous chapter.

4.5 The case of odd N

We now turn to the case where $N = 2M + 1$, $M \geq 1$. We have the following theorem.

Theorem 4.3. *Suppose for integer $M \geq 1$, $n > N = 2M + 1$, and $z_1 \in \mathbb{R}^N$ is a unit norm vector. Consider an orthogonal $Q \in \mathbb{R}^{N \times N}$ as in (4.11) where $T \in C^{N \times N}$ is unitary and $\Lambda \in C^{N \times N}$ is diagonal with all elements on the unit circle. Then Q satisfies (4.4) and the set of vectors $\{z_1, Qz_1, \cdots, Q^{n-1}z_1\}$ do not lie on an $(N-1)$-dimensional iff the following hold.*

(a) Each element of $T^H z_1$ has magnitude $1/\sqrt{N}$.

(b) There exist real $\theta_1, \cdots, \theta_M$, such that $2M$ of the $2M+1$ diagonal elements of Λ are $e^{\pm j\theta_i}$, are distinct and appear in conjugate pairs. The remaining diagonal element of Λ is -1.

(c) For all $i \in \{1, \cdots, M\}$, $e^{\pm j\theta_i} \neq \pm 1$.

(d) For all $\{i, j\} \subset \{1, \cdots, M\}$, including $i = j$, $\theta_i \pm \theta_j$ are integer multiples of $2\pi/n$.

(e) For all $i \in \{1, \cdots, M\}$, $(\pm\theta_i - \pi)$ are integer multiples of $2\pi/n$.

Proof. As before we first prove necessity. The necessity of (b) and (c) follows from Lemma 4.1. Define w as in (4.12). Again (4.4) is equivalent to (4.13). Consequently, as in the proof of Theorem 4.2, (4.4) implies (a).

The off diagonal elements on the left hand side of (4.13) are now of three types. First for suitable $\{k,l\} \subset \{1,\cdots,N\}$, $k \neq l$, and $\{r,s\} \subset \{1,\cdots,M\}$, $r \neq s$, obey (4.14). The second for suitable suitable $\{k,l\} \subset \{1,\cdots,N\}$, $k \neq l$, and $r \in \{1,\cdots,M\}$, obey (4.15). Thus as in the proof of Theorem 4.2 (4.4) implies (d).

The third type of off diagonal element is for suitable suitable $\{k,l\} \subset \{1,\cdots,N\}$, $k \neq l$, and $r \in \{1,\cdots,M\}$, obey:

$$w_k w_l^* \sum_{i=0}^{n-1} e^{ji(\pm\theta_r - \pi)} = w_k w_l^* \frac{1 - e^{jn(\pm\theta_r - \pi)}}{1 - e^{j(\pm\theta_r - \pi)}}. \tag{4.26}$$

Again because of (c) the denominator is non-zero, and thus for this element to be zero (e) must hold.

To prove sufficiency we first note that to show that (4.4) holds it suffices to show that (4.13) holds. As in the proof of Theorem 4.2 (a-d) suffice to prove that the diagonal elements are n/N and off diagonal elements exemplified by (4.14) and (4.15) are zero. Because of (e) the remaing types of off diagonal elements, i.e. (4.26) are also zero.

It remains to show that $\{z_1, Qz_1, Q^2z_1, \cdots, Q^{n-1}z_1\}$ do not inhabit an $(N-1)$-dimensional hyperplane. To establish a contradiction suppose they do lie on an $(N-1)$-dimensional hyperplane. Then for some $\rho \in \mathbb{R}^N$, $\rho \neq 0$ and all $i,k \subset \{0,\cdots,n-1\}$, (4.16) holds. Consequently with $p^H = \rho' T_n = [p_1^*, \cdots p_N^*] \neq 0$, for all

$i \in \{0, \cdots, n-2\}$, (4.17) holds. Now without sacrificing generality assume that

$$\Lambda = diag\left\{e^{j\theta_1}, e^{-j\theta_1}, e^{j\theta_2}, e^{-j\theta_2}, \cdots, e^{j\theta_M}, e^{-j\theta_M}, -1\right\}. \tag{4.27}$$

As $n > N = 2M + 1$, under (4.27) with

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ e^{j\theta_1} & e^{-j\theta_1} & e^{j\theta_2} & e^{-j\theta_2} & \cdots & e^{j\theta_M} & e^{-j\theta_M} & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ e^{j(N-1)\theta_1} & e^{-j(N-1)\theta_1} & e^{j(N-1)\theta_2} & e^{-j(N-1)\theta_2} & \cdots & e^{j(N-1)\theta_M} & e^{-j(N-1)\theta_M} & (-1)^{(N-1)} \end{bmatrix} \tag{4.28}$$

and

$$f = \begin{bmatrix} p_1^* w_1 \left(1-e^{j\theta_1}\right) \\ p_2^* w_2 \left(1-e^{-j\theta_1}\right) \\ p_3^* w_3 \left(1-e^{j\theta_2}\right) \\ p_4^* w_4 \left(1-e^{-j\theta_2}\right) \\ \vdots \\ p_{N-2}^* w_{N-2} \left(1-e^{j\theta_M}\right) \\ p_{N-1}^* w_{N-1} \left(1-e^{-j\theta_M}\right) \\ 2p_N^* w_N \end{bmatrix} \tag{4.29}$$

there holds:

$$Vf = 0. \tag{4.30}$$

The matrix V is a Vandermonde matrix. Thus because of (b,c), the matrix is nonsingular. Hence for (4.30) to hold, because of (c), for all $i \in \{1, \cdots, N\}$, $p_i w_i = 0$. Because of (a), this must mean that that $p = 0$, establishing a contradiction. $\square$

The foregoing represents a complete characterization of Q when $N = 2M$, and $n > N$. We now show that for every $M \geq 1$, a Q conforming to this characterization can be found. First suppose Λ is as in (4.27). Then with T_e as in (4.20), the $N \times N$ matrix

$$T = diag\left\{T_e, T_e, \cdots, T_e, 1\right\} \tag{4.31}$$

obeys (a) if $z_1 = [1, \cdots, 1]'/\sqrt{N}$.

Indeed in this case under (4.22) one obtains:

$$Q = diag\left\{Q_1, Q_2, \cdots, Q_M, -1\right\}. \tag{4.32}$$

One example of θ_i is to choose for $k \in \{1, \cdots, M\}$

$$\theta_k = \frac{2k\pi}{n} - \pi. \tag{4.33}$$

Clearly these saitisfy (d) and (e). Given that there are M of these with $n > 2M+1$, one has for all $k \in \{1, \cdots, M\}$ that

$$-\pi < \theta_k \leq -\pi + \frac{2M\pi}{n} < 0, \tag{4.34}$$

i.e. these satisfy (b-c) as well. In the case when $M = 1$, this solution conforms to the solution in the previous chapter.

CHAPTER 5

SOURCE LOCALIZATION

In this chapter, we will formulate the optimal source localization problem, provide a necessary and sufficient condition for the optimum solution and then we show how the optimality criteria are met for n>2. Section 5.1 provides a brief introduction. Section 5.2 provides preliminaries including a precise problem statement and its motivation. Specifically it provides three different optimality criteria. Section 5.3 shows that all three criteria have an identical necessary and sufficient condition for optimality. In section 5.4, we show how the conditions are met for the 2D source localization case.

5.1 Introduction

This chapter concerns optimum source localization in two dimensions. We assume that the source to be localized is uniformly distributed in a circle, without loss of generality, centered at the origin. As the source may be hazardous, the sensors localizing it must be placed outside a circle of a larger radius. We will similarly phrase optimality in terms of the Expectation of the Fisher Information Matrix (FIM).

As only statistical information concerning the source location is available the optimization here is phrased in terms of the expectation of a FIM. This contrasts to the source monitoring setting where as the source location is assumed known, the actual FIM rather than its expectation is optimized. Even though this brings additional technical complications, the ultimate solution for sensor placement in both localization and monitoring has broad conceptual similarities.

5.2 Preliminaries

We assume that there are n sensors the i-th located at $x_i \in R^2$. We also assume that the Gaussian random variables w_i are mutually uncorrelated. Straightforward calculations, as shown in section 2.1, reveal that to within a scaling that is independent of y and x_i, the FIM for estimating y from the data s_i, A and β, becomes:

$$\sum_{i=1}^{n} \frac{(x_i - y)(x_i - y)'}{\|x_i - y\|^4}.$$

We assume that y is uniformly distributed on a circle of radius r_1, centered at the origin, i.e. its density obeys:

$$p_Y(y) = \begin{cases} \frac{1}{\pi r_1^2} & \|y\| \leq r_1 \\ 0 & \text{else} \end{cases}. \tag{5.1}$$

Given that y is random, the matrix corresponding to the FIM is its expectation, i.e.

$$F = E\left[\sum_{i=1}^{n} \frac{(x_i - y)(x_i - y)'}{\|x_i - y\|^4}\right]. \tag{5.2}$$

We assume as depicted in figure 5.1, that the x_i lie outside a circle, centered on the origin, of radius greater than r_1. In other words we impose the requirement that for some r_2, there holds:

$$\|x_i\| \geq r_2 > r_1. \tag{5.3}$$

Now, the diagonal elements of the inverse of the FIM provide the Cramer-Rao Lower

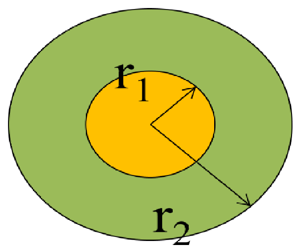

Figure 5.1: Depiction of the localization problem

Bound (CRLB) for the underlying estimation problem conditioned on y. Thus as is typical in detection theory, optimality constitutes either the maximization of the minimum eigenvalue or the determinant of the F, or for reasons laid out below, the minimization of the trace of F^{-1}. Accordingly we pose the three problems below.

Problem 1: For a given integer $n \geq 3$ and $y \in R^2$, find distinct, $x_i \in R^2$, $i \in \{1, \cdots n\}$, such that $\lambda_{\min}(F)$ is maximized subject to (5.3).

Problem 2: For a given integer $n \geq 3$ and $y \in R^2$, find distinct, $x_i \in R^2$, $i \in \{1, \cdots n\}$, such that $\det(F)$ is maximized subject to (5.3).

Problem 3: For a given integer $n \geq 3$ and $y \in R^2$, find distinct, $x_i \in R^2$, $i \in \{1, \cdots n\}$, such that $\operatorname{trace}(F^{-1})$ is minimized subject to (5.3).

Observe, the minimization of $\mathrm{tr}(F^{-1})$ is equivalent to the minimization of the total mean-square error.

5.3 Equivalent Formulation

We will first show that the first inequality constraint in (5.3) is replaceable by an equality constraint, in other words optimum placement can be achieved by placing sensors on the circle centered at the origin and with radius r_2.

The derivation of this fact is somewhat nontrivial. Towards this end consider a summand in (5.2), i.e.:

$$H(x) = E\left[\frac{(x-y)(x-y)'}{\|x-y\|^4}\right]. \tag{5.4}$$

We first make an important observation about $H(x)$.

Lemma 5.1. *Consider $H(x)$ as in (5.4), with $x, y \in R^N$, $N > 1$, y uniformly distributed with pdf as in (5.1) and with*

$$\|x\| \geq r_2 > r_1. \tag{5.5}$$

Then with $\Omega \in \mathbb{R}^{N\times N}$ an orthogonal matrix, there holds:

$$H(\Omega x) = \Omega H(x)\Omega'.$$

Proof. Define $Z = \Omega' Y$. Then because of (5.1) and the fact that Ω is orthogonal, we obtain:

$$p_Z(z) = \begin{cases} \frac{1}{\pi r_1^2} & \|z\| \leq r_1 \\ 0 & \text{else} \end{cases}.$$

Then:

$$\begin{aligned} H(\Omega x) &= E\left[\frac{(\Omega x - y)(\Omega x - y)'}{\|\Omega x - y\|^4}\right] \\ &= \Omega E\left[\frac{(x - z)(x - z)'}{\|x - z\|^4}\right]\Omega' \\ &= \Omega E\left[\frac{(x - y)(x - y)'}{\|x - y\|^4}\right]\Omega' \\ &= \Omega H(x)\Omega'. \end{aligned}$$

□

An important property of $H(x)$ is proved below.

Lemma 5.2. *Consider $H(x)$ as in (5.4), with $x, y \in R^2$, y uniformly distributed with pdf as in (5.1) and under (5.5). Then $H(x)$ is positive definite.*

Proof. In view of Lemma 5.1 to prove positive definiteness it suffices to show that for all $R > r_1$, $H([R, 0]')$ is positive definite. Equivalently we need to show that for all real α, and $R > r_1$,

$$\begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} H([R, 0]') \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} > 0. \tag{5.6}$$

By definition this is equivalent to showing that the following is positive for all real α, and $R > r_1$,

$$\frac{1}{\pi r_1^2}\int_0^{r_1}\int_0^{2\pi}\frac{[(R-r\cos\theta)\cos\alpha+r\sin\theta\sin\alpha]^2}{\left[(R-r\cos\theta)^2+r^2\sin^2\theta\right]^2}d\theta dr. \tag{5.7}$$

Since the integrand in (5.7) is clearly nonnegative for all real α, θ and $R > r$ and this quantity is analytic in θ, r and R for $R > r$, it suffices to show that for every α it is non-zero for at least one set of values of θ, r and R for $R > r$. This is clearly true for $\theta = 0$ when $\cos\alpha \neq 0$, and $\theta = \pi/2$ when $\cos\alpha = 0$. □

We next prove a pivotal lemma.

Lemma 5.3. *Consider $H(x)$ as in (5.4), with $x, y \in R^2$, and y uniformly distributed with pdf as in (5.1). Suppose*

$$R_1 \geq R_2 > r_1. \tag{5.8}$$

Then there holds:

$$H([R_1, 0]') \leq H([R_2, 0]').$$

Proof. To prove the result it suffices to show that for every real α and $r < R$, the left hand side of (5.6) and hence the following is decreasing in R:

$$\int_0^{2\pi} \frac{\left[(R - r\cos\theta)\cos\alpha + r\sin\theta\sin\alpha\right]^2}{\left[(R - r\cos\theta)^2 + r^2\sin^2\theta\right]^2} d\theta.$$

Through a simple scaling by r if need be, it thus suffices to prove that for every real α and $R > 1$ the following is decreasing in R:

$$G_1 = \int_0^{2\pi} \frac{\left[(R - \cos\theta)\cos\alpha + \sin\theta\sin\alpha\right]^2}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2} d\theta.$$

Now observe that:

$$\begin{aligned} G_1 &= \int_0^{\pi} \left\{ \frac{\left[(R - \cos\theta)\cos\alpha + \sin\theta\sin\alpha\right]^2}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2} \right. \\ &+ \left. \frac{\left[(R - \cos\theta)\cos\alpha - \sin\theta\sin\alpha\right]^2}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2} \right\} d\theta \\ &= 2 \int_0^{\pi} \frac{(R - \cos\theta)^2\cos^2\alpha + \sin^2\theta\sin^2\alpha}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2} d\theta \end{aligned}$$

Now for every θ, and $R > 1$,

$$\frac{1}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2}$$

is a decreasing function of R. Thus it suffices to show that for all $R > 1$,

$$\int_0^{\pi} \frac{(R - \cos\theta)^2}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2} d\theta$$

is a decreasing function of R.

As shown in Lemma A.1 in the appendix this integral is:

$$\frac{\pi\left(2R^2-1\right)}{2\left(R-1\right)^2 R^2}.$$

Its derivative with respect to R, for $R > 1$ is negative iff

$$\begin{aligned} 2R(R-1)\left[2R^2(R-1)-(2R^2-1)(2R-1)\right] &< 0 \\ \Leftrightarrow -2R^3+2R-1 &< 0 \\ \Leftarrow -1 &< 0. \end{aligned}$$

This proves the result. $\square$

We need one last result to prove the main result of this section.

Lemma 5.4. *Consider two symmetric matrices A and B, with equal dimensions, that obey $A \geq B > 0$. Then with $\lambda_{min}(\cdot)$, representing the smallest eigenvalue of its argument, there holds:*

(A) $\lambda_{min}(A) \geq \lambda_{min}(B)$.

(B) $\det(A) \geq \det(B)$.

(C) $trace(A^{-1}) \leq trace(B^{-1})$.

Proof. By assumption there is a symmetric matrix $C \geq 0$, such that $A = B + C$. To prove (A) we observe that for x the eigenvector corresponding to the smallest eigenvalue of A, there holds:

$$\begin{aligned} \lambda_{min}(A)x'x &= x'Ax \\ &\geq x'Bx \\ &\geq \lambda_{min}(B)x'x. \end{aligned}$$

To prove (B) we note that for any symmetric positive semidefinite matrix D, $\det(I+D) \geq 1$. As B is symmetric positive definite, there exists a unique positive definite symmetric square root, $B^{1/2}$ of B, such that $\left\{B^{1/2}\right\}^2 = B$. Observe $D = B^{-1/2}CB^{-1/2}$ is symmetric positive semidefinite. Then the result follows by noting that

$$\begin{aligned} \det(A) &= \det(B+C) \\ &= \det(B)\det(I+D) \\ &\geq \det(B). \end{aligned}$$

Finally (C) is a trivial consequence of the fact that $A \geq B > 0$ implies $B^{-1} \geq A^{-1}$. $\square$

We can now state and prove the main result of this section that sets up a simplifying reformulation.

Theorem 5.5. *For any n and $x_i, y \in R^2$, consider (5.2) under (5.3) and with y*

having pdf as in (5.1). Then there exist $z_i \in R^2$, with $\|z_i\| = r_2$, such that:

$$E\left[\sum_{i=1}^{n} \frac{(x_i - y)(x_i - y)'}{\|x_i - y\|^4}\right] \leq E\left[\sum_{i=1}^{n} \frac{(z_i - y)(z_i - y)'}{\|z_i - y\|^4}\right].$$

Proof. Consider a particular x_i. There exists an orthogonal matrix Ω such that

$$x_i = \Omega \begin{bmatrix} \|x_i\| \\ 0 \end{bmatrix}.$$

Then because of Lemma 5.1, with $H(\cdot)$ defined as in (5.4), there holds:

$$H(x_i) = \Omega H\left([\|x_i\|, 0]'\right) \Omega'.$$

Further, from Lemma 5.3:

$$\begin{aligned} H(x_i) &= \Omega H\left([\|x_i\|, 0]'\right) \Omega' \\ &\leq \Omega H\left([r_2, 0]'\right) \Omega' \\ &= H\left([r_2, 0]'\Omega'\right). \end{aligned}$$

Thus

$$z_i = \Omega \begin{bmatrix} \|r_2\| \\ 0 \end{bmatrix}$$

has norm r_2 and obeys:

$$E\left[\frac{(x_i-y)(x_i-y)'}{\|x_i-y\|^4}\right] \le E\left[\frac{(z_i-y)(z_i-y)'}{\|z_i-y\|^4}\right].$$

Thus corresponding to every $\|x_i\| \ge r_2$, there is a $\|z_i\| = r_2$ for which the above inequality holds. Hence the result follows. □

Now define with $\|z_i\| = r_2$,

$$Z = E\left[\sum_{i=1}^{n}\frac{(z_i-y)(z_i-y)'}{\|z_i-y\|^4}\right]. \tag{5.9}$$

Thus in view of Lemma 5.4, without loss of generality to consider problems 1 to 3 is equivalent to considering the three problems below.

Problem 1A: For a given integer $n \ge 3$, find distinct, $z_i \in R^2$, $i \in \{1, \cdots n\}$, such that $\|z_i\| = r_2$ and $\lambda_{\min}(Z)$ is maximized.

Problem 2A: For a given integer $n \ge 3$, find distinct, $z_i \in R^2$, $i \in \{1, \cdots n\}$, such that $\|z_i\| = r_2$ and $\det(Z)$ is maximized.

Problem 3A: For a given integer $n \ge 3$, find distinct, $z_i \in R^2$, $i \in \{1, \cdots n\}$, such that $\|z_i\| = r_2$ and $\mathrm{tr}(Z^{-1})$ is minimized.

Observe that when the z_i all have norm r_2 then

$$\mathrm{trace}(Z) = \mathrm{trace}\left(H\left([r_2, 0]\right)\right) = c. \tag{5.10}$$

We next present a necessary and sufficient condition for the solutions of all three problems the proof of which is a trivial variation of a result in [11].

Theorem 5.6. *Consider for an integer* $n \geq N+1$ *any* $N \times N$ *symmetric positive definite matrix* B*, with*

$$\mathrm{tr}(B) = c. \tag{5.11}$$

Then

$$\lambda_{\min}(B) \leq \frac{cn}{N} \tag{5.12}$$

$$\det(B) \leq \left(\frac{cn}{N}\right)^N, \tag{5.13}$$

and

$$\mathrm{tr}(B^{-1}) \geq \frac{N^2}{cn}. \tag{5.14}$$

Further equality holds in these inequalities iff

$$B = \frac{cn}{N} I. \tag{5.15}$$

Thus the solution to both problems is characterized by $\|z_i\| = r_2$ that result in

$$Z = \frac{cn}{N} I. \tag{5.16}$$

In the next section, we will explore the design of distinct non-coplanar z_i to force (5.16) with $N = 2$ and $n > 2$.

5.4 Solution to the source localization problem

In this section we describe how (5.16) can be met for $n > 2$. We first note certain simple structural issues associated with (5.9). In particular note that for any orthogonal matrix Q, (5.9) is unaltered if z_i are replaced $\pm Qz_i$. Solutions thus related will be designated henceforth as belonging to the same equivalence class. When we talk of uniqueness of optimizing solutions, we mean uniqueness to within such equivalences.

To fix ideas, assume that $H([r_2, 0]')$ has the singular value decomposition:

$$H([r_2, 0]') = U\Lambda U' \tag{5.17}$$

where U is an orthogonal matrix, and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{5.18}$$

As $H([r_2, 0]')$ is positive definite, the λ_i are positive. The next lemma proves that these eigenvalues are in fact distinct.

Lemma 5.7. *Consider $H(x)$ as in (5.4), with $x, y \in R^2$, y uniformly distributed with pdf as in (5.1) and under (5.5). Then the two eigenvalues of $H(x)$ are distinct.*

Proof. In view of Lemma 5.1 to prove that $H(x)$ has distinct eigenvalues it suffices to show that for all $R > r_1$, $H([R, 0]')$ has distinct eigenvalues. To establish a contradiction suppose that both the eigenvalues of $H([R, 0]')$ are the same. As $H([R, 0]')$ is positive definite this must mean that it is a scaled identity. This in turn implies that for every given $R > r_1$ the integral in (5.7) is constant regardless of α.

Now observe that this is equivalent to stating that for every $R > r_1$ the integral in (5.7) viewed as a function of α, is independent of α. Arguing as in the proof of Lemma 5.3 this is equivalent to saying that for all $R > r_1$, the following integeral is independent of α:

$$\int_0^{r_1} \int_0^{\pi} \frac{\left[(R - r\cos\theta)^2 \cos^2\alpha + r^2 \sin^2\theta \sin^2\alpha\right]}{\left[(R - r\cos\theta)^2 + r^2 \sin^2\theta\right]^2} d\theta dr.$$

Further to prove that this cannot be true, it suffices to show that for all $R > 1$, the derivative with respect to α of the integral below has the same sign as $\sin 2\alpha$. This is so as in that case the derivative of (5.7) must be nonzero for almost all values of α.

$$\int_0^{\pi} \frac{\left[(R - \cos\theta)^2 \cos^2\alpha + \sin^2\theta \sin^2\alpha\right]}{\left[(R - \cos\theta)^2 + \sin^2\theta\right]^2} d\theta.$$

This derivative is simply:

$$\sin 2\alpha \int_0^\pi \frac{\left[(R-\cos\theta)^2 - \sin^2\theta\right]}{\left[(R-\cos\theta)^2 + \sin^2\theta\right]^2} d\theta.$$

Thus to prove the result all we need to show is that for $R > 1$,

$$\int_0^\pi \frac{\left[(R-\cos\theta)^2 - \sin^2\theta\right]}{\left[(R-\cos\theta)^2 + \sin^2\theta\right]^2} d\theta > 0.$$

From Lemmas A.1 and A.3 this requires

$$\begin{aligned}
\frac{(2R^2-1)}{2\,(R-1)^2\,R^2} - \frac{1}{R^4-1} &> 0 \\
\Leftrightarrow \frac{2R^2-1}{2R^2} - \frac{1}{R^2+1} &> 0 \\
\Leftarrow (2R^2-1)(R^2+1) - 1 &> 0 \\
\Leftarrow R^2+1-1 &> 0,
\end{aligned}$$

which is clearly true. $\square$

The fact that the eigenvalues are distinct ensures that the problems we are attempting to solve do not have trivial soultions, as otherwise $H(x)$ and Z are trivially scaled identity matrices.

Observe also that there are two types of 2×2 orthogonal matrices. The first known as Givens rotations take the form:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{5.19}$$

Most specifically, with such a Q, Qx is x rotated counter clockwise by θ. The second type of orthogonal matrix has the form:

$$\bar{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}. \tag{5.20}$$

Observe these two matrices are related to each other by:

$$Q = \bar{Q} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

There is no loss of generality in assuming that U in (5.17) is as in (5.19). This is so as for diagonal Λ

$$\begin{aligned} Q\Lambda Q' &= \bar{Q} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{Q}' \\ &= \bar{Q}\Lambda\bar{Q}'. \end{aligned}$$

Thus we will assume that U is a Givens rotation. We also note an important fact about Givens rotations exploited in the sequel: These matrices commute.

In view of the equivalences noted earlier, there is no loss of generality in assuming that

$$z_1 = U'[r_2, 0]', \tag{5.21}$$

an assumption that will hold henceforth. We next describe one class of potential solutions for achieving (5.16). Namely with Q a matrix as in (5.19), select for $i \in \{2, \cdots, n\}$

$$z_i = Q^{i-1} z_1. \tag{5.22}$$

Then (5.16) is equivalent to:

$$\sum_{i=1}^{n} H\left(Q^{i-1} z_1\right) = \frac{n\left(\lambda_1 + \lambda_2\right)}{2} I. \tag{5.23}$$

We observe also that:

$$Q^i = \begin{bmatrix} \cos i\theta & -\sin i\theta \\ \sin i\theta & \cos i\theta \end{bmatrix}.$$

Then because of Lemma 5.1, (5.17) and (5.21), the left hand side of (5.22) is:

$$\begin{aligned} \sum_{i=1}^{n} H\left(Q^{i-1} z_1\right) &= \sum_{i=1}^{n} Q^{i-1} U' H\left([r_2, 0]'\right) U \left(Q^{i-1}\right)' \\ &= \sum_{i=1}^{n} Q^{i-1} \Lambda \left(Q^{i-1}\right)' \end{aligned}$$

We now assert that in (5.19)

$$\theta = \frac{\pi}{n} \tag{5.24}$$

suffices to achieve (5.22). Indeed observe that under this choice the $(1,1)$ element of the left hand side of (5.23) is

$$Z(1,1) = \lambda_1 \left(1 + \sum_{i=1}^{n-1} \cos^2 \frac{\pi i}{n}\right) + \lambda_2 \sum_{i=1}^{n-1} \sin^2 \frac{\pi i}{n}, \tag{5.25}$$

the $(2,2)$ element is:

$$Z(2,2) = \lambda_2 \left(1 + \sum_{i=1}^{n-1} \sin^2 \frac{\pi i}{n}\right) + \lambda_1 \sum_{i=1}^{n-1} \cos^2 \frac{\pi i}{n}, \tag{5.26}$$

and the off diagonal elements are:

$$Z(1,2) = (\lambda_1 - \lambda_2) \sum_{i=1}^{n-1} \cos \frac{\pi i}{n} \sin \frac{\pi i}{n}. \tag{5.27}$$

Now observe that (5.27) becomes:

$$\begin{aligned} Z(1,2) &= (\lambda_1 - \lambda_2) \sum_{i=1}^{n-1} \cos \frac{\pi i}{n} \sin \frac{\pi i}{n} \\ &= \frac{\lambda_1 - \lambda_2}{2} \sum_{i=1}^{n-1} \sin \frac{2\pi i}{n} \\ &= (\lambda_1 - \lambda_2)\, \mathrm{Im} \left[\sum_{i=0}^{n-1} \exp(j2\pi i/n) \right] \\ &= 0. \end{aligned}$$

Further,

$$
\begin{aligned}
Z(1,1) &= \lambda_1 + \lambda_1 \sum_{i=1}^{n-1} \cos^2 \frac{\pi i}{n} + \lambda_2 \sum_{i=1}^{n-1} \sin^2 \frac{\pi i}{n} \\
&= \lambda_1 + (n-1)\lambda_2 + (\lambda_1 - \lambda_2) \sum_{i=1}^{n-1} \cos^2 \frac{\pi i}{n} \\
&= \lambda_1 + (n-1)\lambda_2 + \frac{\lambda_1 - \lambda_2}{2} \sum_{i=1}^{n-1} \left(1 + \cos \frac{2\pi i}{n}\right) \\
&= \lambda_1 + (n-1)\lambda_2 + \frac{(n-1)(\lambda_1 - \lambda_2)}{2} \\
&+ \frac{\lambda_1 - \lambda_2}{2} \sum_{i=1}^{n-1} \cos \frac{2\pi i}{n} \\
&= \lambda_1 + (n-1)\lambda_2 + \frac{(n-1)(\lambda_1 - \lambda_2)}{2} - \frac{\lambda_1 - \lambda_2}{2} \\
&+ \frac{\lambda_1 - \lambda_2}{2} \sum_{i=0}^{n-1} \cos \frac{2\pi i}{n} \\
&= \lambda_1 + (n-1)\lambda_2 + \frac{(n-2)(\lambda_1 - \lambda_2)}{2} - \frac{\lambda_1 - \lambda_2}{2} \\
&+ \frac{\lambda_1 - \lambda_2}{2} \operatorname{Re} \left[\sum_{i=0}^{n-1} e^{j\frac{2\pi i}{n}} \right] \\
&= \lambda_1 + (n-1)\lambda_2 + \frac{(n-2)(\lambda_1 - \lambda_2)}{2} - \frac{\lambda_1 - \lambda_2}{2} \\
&= \frac{n}{2}(\lambda_1 + \lambda_2).
\end{aligned}
$$

Then by symmetry one also has

$$
Z(2,2) = \frac{n}{2}(\lambda_1 + \lambda_2),
$$

and hence (5.22).

Thus a set of n vectors each rotated from their neighbor by π/n, and their equivalences suffice to achieve (5.16). It should be stressed that this choice of z_i is by no means unique for $n > 3$. We now argue that it is unique for $n = 3$, within of course the equivalences noted above. Indeed for $n = 3$, to within equivalences we must choose z_1 as in (5.21), and for $i \in \{1, 2\}$, $z_{i+1} = Q_i z_1$, where

$$Q_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}.$$

Then we must have:

$$\lambda_1 \left(1 + \cos^2\theta_1 + \cos^2\theta_2\right) + \lambda_2 \left(\sin^2\theta_1 + \sin^2\theta_2\right) = \frac{3\left(\lambda_1 + \lambda_2\right)}{2}, \tag{5.28}$$

$$\lambda_2 \left(1 + \cos^2\theta_1 + \cos^2\theta_2\right) + \lambda_1 \left(\sin^2\theta_1 + \sin^2\theta_2\right) = \frac{3\left(\lambda_1 + \lambda_2\right)}{2}, \tag{5.29}$$

and

$$(\lambda_1 - \lambda_2)\left(\cos\theta_1 \sin\theta_1 + \cos\theta_2 \sin\theta_2\right) = 0. \tag{5.30}$$

Since the λ_i are distinct, (5.30) is equivalent to:

$$\cos\theta_1 \sin\theta_1 + \cos\theta_2 \sin\theta_2 = 0. \tag{5.31}$$

Subtracting λ_1 times (5.29) from λ_2 times (5.28) one gets, using the distinct nature of the λ_i that:

$$\sin^2\theta_1 + \sin^2\theta_2 = 3/2. \tag{5.32}$$

Similarly, subtracting λ_1 times (5.28) from λ_2 times (5.29) one gets, using the distinct nature of the λ_i that:

$$1 + \cos^2\theta_1 + \cos^2\theta_2 = 3/2. \tag{5.33}$$

It has been shown in [21] that to within equivalences the solution to this is unique, i.e. $\theta_2 = 2\theta_1 = 2\pi/3$.

CHAPTER 6

CONCLUSION

The problem of source localization has become increasingly important in recent years. We are interested in estimating the location of a source using various relative position information. This research considered source localization and source monitoring using relative position information provided by Received Signal Strength (RSS) values under the assumption of log-normal shadowing.

Two specific issues were investigated. The first was one of source monitoring. In this, one must place sensors around a localized source in an optimum fashion subject to the constraint that sensors were at least a certain distance from the source. The second was sensor placement for source localization. In this problem, we assumed that the source was uniformly distributed in a circular region. The sensors must be placed in the complement of a larger concentric circle, to optimally localize the source.

In the source monitoring problem, we considered the optimum placement of sensors in two and three dimension scenarios. The problem statement became one of investigating Fisher Information Matrix (FIM) evaluation functions. In the 2D case, the determinant and minimum eigenvalue of FIM were computed while in the 3D case an additional evaluation function, the trace of the inverse FIM was also computed. The underlying problem then became one of maximizing the determinant or the minimum eigenvalue of FIM or minimizing the trace of the inverse FIM (3D case). Optimality of sensor placement for source monitoring was achieved if and only if the Fisher Information Matrix became a scaled diagonal matrix. We have provided means by which sensors can be placed to achieve this condition in 2, 3 and arbitrary N dimensions.

In the source localization problem, we also considered optimum placement of non-colinear sensors in two dimensions for maximizing the determinant and the smallest eigenvalue of the expectation of the FIM associated with the localization from RSS under log normal shadowing, or by minimizing the trace of the inverse of the expectation of the FIM. Optimality, has been subject to the requirement that the source was uniformly distributed inside a circle of radius r_1, and that the sensors be outside a larger radius concentric circle. We have shown that for optimality, it was necessary and sufficient that the expectation of the FIM be a scaled identity matrix. We have thus, provided a class of locations that achieve optimality.

Future work includes extending the localization problem to 3-dimensions. It should be also interesting to consider different geometries, e.g, when the source is in a rectangle or an oval, and the sensors must be placed on its perimeters.

Other forms of relative position information such as TDOA, TOA or angle of arrival should also be considered.

APPENDIX A

SELECTED LEMMAS

We evaluate here two integrals used in the text of the section.

Lemma A.1. *For all $R > 1$ there holds:*

$$\int_0^\pi \frac{(R-\cos(\theta))^2}{\left[(R-\cos(\theta))^2+\sin^2(\theta)\right]^2} d\theta = \frac{\pi\left(2R^2-1\right)}{2\left(R-1\right)^2 R^2}.$$

Proof. Define $t = \tan\left(\frac{\theta}{2}\right)$. Then $\theta = 2\tan^{-1}(t)$,

$$d\theta = \frac{2}{1+t^2} dt,$$

and $\cos(\theta) = \frac{1-t^2}{1+t^2}$, where Thus:

$$\int_0^\pi \frac{(R-\cos(\theta))^2}{\left(R^2-2R\cos(\theta)+1\right)^2} d\theta$$

$$= \int_0^\infty \frac{2}{1+t^2} \frac{\left(R-\left(\frac{1-t^2}{1+t^2}\right)\right)^2}{\left(R^2-2R\left(\frac{1-t^2}{1+t^2}\right)+1\right)^2} dt$$

$$= \int_0^\infty \frac{2}{1+t^2}\left(\frac{R\left(1+t^2\right)-\left(1-t^2\right)}{R^2\left(1+t^2\right)-2R\left(1-t^2\right)+\left(1+t^2\right)}\right)^2 dt$$

$$= \int_0^\infty \frac{2}{1+t^2} \left(\frac{(R-1) + t^2 (R+1)}{(R-1)^2 + t^2 (R+1)^2} \right)^2 dt$$

$$= 2 \int_0^\infty \frac{t^4 (R+1)^2 + 2t^2 (R^2-1) + (R-1)^2}{(1+t^2) \left[(R-1)^2 + t^2 (R+1)^2 \right]^2} dt$$

Let $\frac{R+1}{R-1} = V$, then the integral becomes

$$\frac{2}{(R-1)^2} \int_0^\infty \frac{t^4 V^2 + 2t^2 V + 1}{(1+t^2) \left[1 + t^2 V^2 \right]^2} dt \tag{A.1}$$

Using partial fractions, this reduces to:

$$\frac{2}{(R-1)^2} \int_0^\infty \Bigg[\frac{V-1}{(V+1)(V^2 t^2 + 1)^2} + \frac{1}{(V+1)^2 (t^2+1)}]$$

$$+ \frac{2V+1}{(V+1)^2 (V^2 t^2 + 1)} \Bigg] dt$$

Following two integrals are well known or easily derived:

$$\int \frac{dt}{1 + V^2 t^2} = \frac{\tan^{-1}(Vt)}{V} + C, \tag{A.2}$$

and

$$\int \frac{dt}{\left(1+V^2t^2\right)^2} = \frac{\tan^{-1}(Vt)}{2V} + \frac{t}{2\left(1+V^2t^2\right)} + C. \tag{A.3}$$

Thus the integral becomes:

$$\frac{\pi}{\left(R-1\right)^2}\left[\frac{V-1}{2V(V+1)} + \frac{1}{(V+1)^2} + \frac{2V+1}{V(V+1)^2}\right]$$

$$= \frac{\pi\left(V^2+6V+1\right)}{2\left(R-1\right)^2\left(V+1\right)^2 V}$$

$$= \frac{\pi\left((R+1)^2+6(R^2-1)+(R-1)^2\right)}{8\left(R-1\right)^2 R^2}$$

$$= \frac{\pi\left(8R^2-4\right)}{8\left(R-1\right)^2 R^2}$$

$$= \frac{\pi\left(2R^2-1\right)}{2\left(R-1\right)^2 R^2}$$

□

Lemma A.2.

$$\frac{d}{dR}\left(\frac{\pi\left(2R^2-1\right)}{2\left(R-1\right)^2R^2}\right)=-\frac{\pi\left(2R^3-2R+1\right)}{\left(R-1\right)^3R^3}$$

Proof.

$$\frac{d}{dR}\left(\frac{\pi\left(2R^2-1\right)}{2\left(R-1\right)^2R^2}\right)=\frac{\pi}{2}\frac{d}{dR}\left(\frac{2R^2-1}{\left(R-1\right)^2R^2}\right)$$

Using Product rule

$$\frac{d\left(UV\right)}{dR}=\frac{dU}{dR}V+U\frac{dV}{dR}$$

where $U=\frac{1}{(R-1)^2}$ and $V=\frac{2R^2-1}{R^2}$.

$$\frac{d}{dR}\left(\frac{\pi\left(2R^2-1\right)}{2\left(R-1\right)^2R^2}\right)=\frac{\pi}{2}\left(\frac{2R^2-1}{R^2}\frac{d}{dR}\left(\frac{1}{\left(R-1\right)^2}\right)+\frac{1}{\left(R-1\right)^2}\frac{d}{dR}\left(\frac{2R^2-1}{R^2}\right)\right)$$

Using the chain rule

$$\frac{dU^n}{dR}=nU^{n-1}\frac{dU}{dR}$$

where $U=R-1$ and $n=-2$.

$$=\frac{\pi}{2}\left(\frac{1}{\left(R-1\right)^2}\frac{d}{dR}\left(\frac{2R^2-1}{R^2}\right)-\frac{2\left(2R^2-1\right)}{\left(R-1\right)^3R^2}\frac{d}{dR}\left(R-1\right)\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{(R-1)^2}\frac{d}{dR}\left(\frac{2R^2-1}{R^2}\right) - \frac{2\left(2R^2-1\right)}{(R-1)^3 R^2}\right)$$

Using the product rule

$$= \frac{\pi}{2}\left(\frac{\left(2R^2-1\right)\frac{d}{dR}\left(\frac{1}{R^2}\right) + \frac{1}{R^2}\frac{d}{dR}\left(2R^2-1\right)}{(R-1)^2} - \frac{2\left(2R^2-1\right)}{(R-1)^3 R^2}\right)$$

$$= \frac{\pi}{2}\left(\frac{\frac{4}{R} - \frac{2\left(2R^2-1\right)}{R^3}}{(R-1)^2} - \frac{2\left(2R^2-1\right)}{(R-1)^3 R^2}\right)$$

Simplifying the above equation yields

$$\frac{\pi}{2}\frac{d}{dR}\left(\frac{2R^2-1}{(R-1)^2 R^2}\right)$$

$\square$

Lemma A.3. *For all $R > 1$ there holds:*

$$\int_0^{\pi} \frac{\sin^2\theta}{\left[(R-\cos(\theta))^2 + \sin^2(\theta)\right]^2} d\theta = \frac{\pi}{R^4-1}.$$

Proof. Proceed as in the proof of Lemma A.1. With t and V as defined, using (A.2) and (A.3) one obtains:

$$\int_0^{\pi} \frac{\sin^2\theta}{\left(R^2 - 2R\cos(\theta) + 1\right)^2} d\theta$$

$$= \int_0^\infty \frac{8t^2}{(1+t^2)\left(R^2 - 2R\left(\frac{1-t^2}{1+t^2}\right) + 1\right)^2} dt$$

$$= \frac{8}{(R-1)^4} \int_0^\infty \frac{t^2}{(1+t^2)\left[(R-1)^2 + t^2(R+1)^2\right]^2} dt$$

$$= \frac{8}{(R-1)^4} \int_0^\infty \frac{t^2}{(1+t^2)\left[1+t^2V^2\right]^2} dt \tag{A.4}$$

$$= \frac{8}{(R-1)^4} \int_0^\infty \left[\frac{1}{(1-V^2)(V^2t^2+1)^2}\right.$$

$$\left. - \frac{1}{(V^2-1)^2(t^2+1)} + \frac{V^2}{(V^2-1)^2(V^2t^2+1)}\right] dt$$

$$= \frac{4\pi}{(R-1)^4}\left[\frac{V-1}{(V^2-1)^2} + \frac{1}{2V(1-V^2)}\right]$$

$$= \frac{2\pi}{(R-1)^4 V(V^2+1)}$$

$$= \frac{2\pi}{(R-1)^4 \frac{R+1}{R-1}\left(\left(\frac{R+1}{R-1}\right)^2 + 1\right)}$$

$$= \frac{\pi}{(R^2-1)(R^2+1)}$$

proving the result. $\square$

REFERENCES

[1] *IEEE Signal Processing Magazine*, July 2005.

[2] T. Ajdler, I. Kozintsev, R. Lienhart, and M. Vetterli. Acoustic source localization in distributed sensor networks. In *Signals, Systems and Computers, 2004. Conference Record of the Thirty-Eighth Asilomar Conference on*, volume 2, pages 1328–1332 Vol.2, Nov. 2004.

[3] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan. *Estimation with Applications to Tracking and Navigation.* John Wiley, New York, 2001.

[4] D. Blatt and A.O. Hero. Energy-based sensor network source localization via projection onto convex sets. *Signal Processing, IEEE Transactions on*, 54(9):3614–3619, Sept. 2006.

[5] M. Cao, B. D.O. Anderson, and A. S. Morse. Sensor network localization with imprecise distances. *Systems & Control Letters*, 55(11):887–893, November 2006.

[6] G. C. Carter. Time-delay estimation for passive sonar processing. *IEEE Trans. Acoust., Speech, Signal Process*, 29:463–470, 1981.

[7] J.C. Chen, Kung Yao, and R.E. Hudson. Source localization and beamforming. *Signal Processing Magazine, IEEE*, 19(2):30–39, Mar 2002.

[8] J. Cortes, S. Martinez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. In *Robotics and Automation, 2002. Proceedings. ICRA '02. IEEE International Conference on*, volume 2, pages 1327–1332, 2002.

[9] S. H. Dandach and F. Bullo. Algorithms for regional source localization. In *acc*, pages 5440–5445, St. Louis, MO, June 2009.

[10] S.H. Dandach, B. Fidan, S. Dasgupta, and B.D.O. Anderson. Adaptive source localization by mobile agents. In *Decision and Control, 2006 45th IEEE Conference on*, pages 2045–2050, Dec. 2006.

[11] S. Dasgupta, S. R. Ibeawuchi, and Z. Ding. Optimum sensor placement for source monitoring under log-normal shadowing. In *Proceedings of IFAC Workshop on System Identification*, July 2009.

[12] Soura Dasgupta, Stella-Rita C. Ibeawuchi, and Zhi Ding. Optimum sensor placement for source monitoring under log-normal shadowing in three dimensions. In *ISCIT'09: Proceedings of the 9th international conference on Communications*

and information technologies, pages 376–381, Piscataway, NJ, USA, 2009. IEEE Press.

[13] B. Fidan, S. Dasgupta, and B.D.O. Anderson. Guaranteeing practical convergence in algorithms for sensor and source localization. *Signal Processing, IEEE Transactions on*, 56(9):4458–4469, Sept. 2008.

[14] George H. Forman and John Zahorjan. The challenges of mobile computing. *IEEE Computer*, 27:38–47, 1994.

[15] M. Gavish and A.J. Weiss. Performance analysis of bearing-only target location algorithms. *Aerospace and Electronic Systems, IEEE Transactions on*, 28(3):817–828, Jul 1992.

[16] D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. O. Anderson, P. N. Belhumeur, J. Aspnes, and T. Eren. A theory of network localization j. aspnes, t. eren, 2006.

[17] Gene H. Golub and Charles F. Van Loan. *Matrix Computation*. John Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, Maryland, third edition, 1996.

[18] Tamir Hegazy and George J. Vachtsevanos. Sensor placement for isotropic source localization. In *IPSN*, pages 432–441, 2003.

[19] K. C. Ho, Xiaoning Lu, and L. Kovavisaruch. Source localization using tdoa and fdoa measurements in the presence of receiver location errors: Analysis and solution. *Signal Processing, IEEE Transactions on*, 55(2):684–696, Feb. 2007.

[20] A. Host-Madsen. On the existence of efficient estimators. *Signal Processing, IEEE Transactions on*, 48(11):3028–3031, Nov 2000.

[21] S.R.C. Ibeawuchi, S. Dasgupta, Cheng Meng, and Zhi Ding. Source localization using a maximum likelihood/semidefinite programming hybrid. In *Sensing Technology, 2008. ICST 2008. 3rd International Conference on*, pages 585–588, 30 2008-Dec. 3 2008.

[22] Jason T. Isaacs, Daniel J. Klein, and João Pedro Hespanha. Optimal sensor placement for time difference of arrival localization. In *Proc. of the 48th Conf. on Decision and Contr.*, Dec. 2009.

[23] Chris Savarese Jan and Jan M. Rabaey. Locationing in distributed ad-hoc wireless sensor networks. In *ICASSP*, pages 2037–2040, 2001.

[24] Damien B. Jourdan and Nicholas Roy. Optimal sensor placement for agent localization. *ACM Trans. Sen. Netw.*, 4(3):1–40, 2008.

[25] Brad Karp and H. T. Kung. Gpsr: greedy perimeter stateless routing for wireless networks. In *MobiCom '00: Proceedings of the 6th annual international conference on Mobile computing and networking*, pages 243–254, New York, NY, USA, 2000. ACM.

[26] L. Kovavisaruch and K.C. Ho. Alternate source and receiver location estimation using tdoa with receiver position uncertainties. In *Acoustics, Speech, and Signal Processing, 2005. Proceedings. (ICASSP '05). IEEE International Conference on*, volume 4, pages iv/1065–iv/1068 Vol. 4, March 2005.

[27] Xiaoning Lu and K. C. Ho. Taylor-series technique for moving source localization in the presence of sensor location errors. In *ISCAS*, 2006.

[28] A. W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Applications*. Academic Press, 1979.

[29] S. Martínez and F. Bullo. Optimal sensor placement and motion coordination for target tracking. *Automatica*, 42(4):661–668, 2006.

[30] Chen Meng, Zhi Ding, and S. Dasgupta. A semidefinite programming approach to source localization in wireless sensor networks. *Signal Processing Letters, IEEE*, 15:253–256, 2008.

[31] L. E. Miller, P. F. Wilson, N. P. Bryner, M. H. Francis, J. R. Guerrieri, D. W. Stroup, and L. Klein-berndt. Rfid-assisted indoor localization and communication for first responders. In *In Proc. of the Int. Symposium on Advanced Radio Technologies*, 2006.

[32] B. Mungamuru and P. Aarabi. Enhanced sound localization. *Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on*, 34(3):1526–1540, June 2004.

[33] J. Neering, M. Bordier, and N. Maizi. Optimal passive source localization. In *Sensor Technologies and Applications, 2007. SensorComm 2007. International Conference on*, pages 295–300, Oct. 2007.

[34] N. Patwari, J.N. Ash, S. Kyperountas, III Hero, A.O., R.L. Moses, and N.S. Correal. Locating the nodes: cooperative localization in wireless sensor networks. *Signal Processing Magazine, IEEE*, 22(4):54–69, July 2005.

[35] M.G. Rabbat and R.D. Nowak. Decentralized source localization and tracking [wireless sensor networks]. In *Acoustics, Speech, and Signal Processing, 2004. Proceedings. (ICASSP '04). IEEE International Conference on*, volume 3, pages iii–921–4 vol.3, May 2004.

[36] Michael Rabbat and Robert Nowak. Distributed optimization in sensor networks. In *Proceedings of the third international symposium on Information processing in sensor networks (IPSN '04)*, pages 20–27, Berkeley, California, 2004.

[37] T. S. Rappaport. *Wireless Communications: Principles & Practice*. Prentice Hall, 1996.

[38] T.S. Rappaport, J.H. Reed, and B.D. Woerner. Position location using wireless communications on highways of the future. *Communications Magazine, IEEE*, 34(10):33–41, Oct 1996.

[39] M. Rydstrom, E.G. Strom, and A. Svensson. Robust sensor network positioning based on projections onto circular and hyperbolic convex sets (pocs). In *Signal Processing Advances in Wireless Communications, 2006. SPAWC '06. IEEE 7th Workshop on*, pages 1–5, July 2006.

[40] A.H. Sayed, A. Tarighat, and N. Khajehnouri. Network-based wireless location: challenges faced in developing techniques for accurate wireless location information. *Signal Processing Magazine, IEEE*, 22(4):24–40, July 2005.

[41] X. Sheng and Y. H. Hu. Collaborative source localization in wireless sensor network system. In *GlobeCom*, 2003.

[42] Xiaohong Sheng and Yu hen Hu. Energy based acoustic source localization. In *Proc. IPSN, 2003*, pages 285–300, 2003.

[43] H. L. Van Trees. *Detection, Estimation and Modulation Theory*. Wiley, New York, 1971.

[44] S. Venkatesh and W.C. Karl. A semidefinite programming approach to source localization. In *Signals, Systems and Computers, 2003. Conference Record of the Thirty-Seventh Asilomar Conference on*, volume 2, pages 1943–1947 Vol.2, Nov. 2003.

[45] Mingfei Wang, Linlin Ci, Ping Zhan, and Yongjun Xu. Acoustic source localization in wireless sensor networks. *Intelligent Information Technology Applications, 2007 Workshop on*, 0:196–199, 2007.

[46] E. Weinstein. Optimal source localization and tracking from passive array measurements. *Acoustics, Speech and Signal Processing, IEEE Transactions on*, 30(1):69–76, Feb 1982.

[47] Mark Weiser. Some computer science issues in ubiquitous computing. *Commun. ACM*, 36(7):75–84, 1993.

[48] B. Yang and J. Scheuing. Cramer-rao bound and optimum sensor array for source localization from time differences of arrival. In *Acoustics, Speech, and Signal Processing, 2005. Proceedings. (ICASSP '05). IEEE International Conference on*, volume 4, pages iv/961–iv/964 Vol. 4, March 2005.

[49] Hong Zhang. Two-dimensional optimal sensor placement. *Systems, Man and Cybernetics, IEEE Transactions on*, 25(5):781–792, May 1995.